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SEQUENTIAL VS. SINGLE-ROUND UNIFORM-PRICE AUCTIONS[1]

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Abstract

We study sequential and single-round uniform-price auctions with affiliated values. We derive symmetric equilibrium for the auction in which k_1 objects are sold in the first round and k_2 in the second round, with and without revelation of the first-round winning bids. We demonstrate that auctioning objects in sequence generates a *lowballing effect* that reduces first-round revenue. Thus, revenue is greater in a single-round, uniform auction for $k = k_1 + k_2$ objects than in a sequential uniform auction with no bid announcement. When the first-round winning bids are announced, we also identify two *informational effects*: a positive effect on second-round price and an ambiguous effect on first-round price. The expected first-round price can be greater or smaller than with no bid announcement, and greater or smaller than the expected price in a single-round uniform auction. As a result, total expected revenue in a sequential uniform auction with winning-bids announcement can be greater or smaller than in a single-round uniform auction.

Journal of Economic Literature Classification Numbers: D44, D82.

Keywords: Multi-Unit Auctions, Sequential Auctions, Uniform-Price Auction, Affiliated Values, Information Revelation.

1 Introduction

Uniform-price auctions are widely used to sell identical or quite similar objects. Sometimes sellers auction all objects together in a single round, while other times they auction them separately in a sequence of rounds. For example, cattle, fish, vegetables, timber, tobacco, wine, and frequency transmission rights typically are sold in sequence, while government

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securities and mineral rights are sold in a single round. When using sequential auctions, the seller must decide what information to release after each round of bidding.

Two important questions arise. Do sequential sales raise the seller's revenue, or is revenue maximized in a single-round auction? How do equilibrium prices in each round of a sequential auction depend on the information that the seller reveals about bidding in earlier rounds?

To address these questions, we suppose that the seller owns k identical objects. Each buyer demands only one object.[1] Buyers' value estimates, or signals, are affiliated random variables, as in Milgrom and Weber (1982, 2000). We compare the single-round uniform-price auction and the two-round (or sequential) uniform-price auction, in which k_1 objects are sold in the first round and $k_2 = k - k_1$ are sold in the second round. In both auctions, the price in a given round is equal to the highest losing bid in that round. (The extension of our results to more than two rounds is discussed in Section 4.) For sequential auctions, we focus on two information policies. Under the first policy, the seller does not reveal any information after the first round. Under the second policy, the seller announces all the first-round winning bids before the second round. As we shall see in Section 5.2, intermediate information policies like the policy of revealing only the lowest first-round winning bid - an approximation of the policy of revealing the winning price - yield the seller a lower revenue than the policy of revealing all winning bids.

First, we derive symmetric equilibrium bidding strategies. While equilibrium bidding strategies and seller's revenue in the single-round uniform auction are well known (e.g., see Milgrom and Weber, 2000), we are the first to study and provide an equilibrium for the sequential, uniform-price auction with winning-bids announcement.

The equilibrium of the sequential auction with no bid announcement (when only one object is sold in each round) was conjectured by Milgrom and Weber (2000), first circulated as a working paper in 1982. In a forward and bracketed comments, Milgrom and Weber (2000) explain that the delay in publishing their work was due to the proofs of this and other related results having "refused to come together" (p. 179). They also explain the nature of

[1] Government-run auctions often limit each bidder to bid for at most one asset. This has been the case, for example, in the auctions of national frequency licenses in many European countries in the last few years.

the difficulty, to which we return in Section 4, and add that the conjectured equilibrium "should be regarded as being in doubt" (p. 188). We have been able to prove (for the case of two rounds) that the equilibrium conjectured by Milgrom and Weber (extended to more than just one object per round) is indeed an equilibrium.

According to the linkage principle, public revelation of information raises revenue in the single-item, single-round, symmetric, affiliated-values model (see Milgrom and Weber, 1982). Even if the winning bids are not announced, bidders have additional knowledge after the first round of a sequential auction. They know that at least k_1 bidders bid higher than them. Intuition derived from the linkage principle has lead Milgrom and Weber's (2000) to conjecture that both versions of the sequential auction raise greater revenue than a single-round auction for k objects (see also fn. 5). We show that this conjecture is incorrect.

In the symmetric equilibrium of a sequential uniform auction with no bid announcement, the new information available to bidders after the first round turns out to be useless. In the second round, the bidding function is as in a single-round uniform auction for k objects, because in the symmetric equilibrium of a single-round uniform auction bidders bid as if they are tied with the k-th highest bid; that is, they already assume that there are $k-1 \geq k_1$ bidders with signals higher than theirs. Auctioning objects in sequence, however, has a cost to the seller. It induces bidders to lowball in the first round by making bids that are conditional on being tied with the price setter. This *lowballing effect* reduces first-round revenue. Thus, revenue is greater in a single-round auction than in a sequential auction with no bid announcement.

While we are not the first to show that the linkage principle may not hold in multi-unit auctions, we must stress a very important difference with other examples in the literature that contradict the linkage principle. In all the other examples we are aware of (e.g., see Perry and Reny, 1999, and example 8.2 in Krishna, 2002) either bidders have multi-unit demands, or they are asymmetric; the linkage principle fails because public revelation of information changes the final allocation of the goods. In our model bidders are symmetric, they have unit demands, and both the single-round and the sequential auction are efficient - the winners are the bidders with the k highest signals. The linkage principle fails because in the sequential uniform auction with no bid announcement all bidders lowball in the first round.

In our model, when the first-round winning bids are announced, there is a positive *informational effect* on second-round bids. This effect is closely related to the linkage principle. However, the fact that the first-round winning bids will be revealed also has an effect on first-round bids, which may raise or decrease first-round revenue, depending on the model parameters. The first-round price when the winning bids are announced could be greater or smaller than with no bid announcement, and greater or smaller than the price in a single-round auction. As a result, total revenue in a sequential auction with winning-bids announcement could be greater or smaller than in a single-round auction.

Milgrom and Weber (1982, 2000) have shown that the ascending (English) auction raises the highest revenue among standard single-round auctions, and that the symmetric equilibrium of the sequential ascending auction coincides with the symmetric equilibrium of the single-round ascending auction. As an ascending auction moves forward and bidders drop out, the losers' (i.e., the lowest) signals become known. In a sequential auction with winning-bids announcement, it is the winners' (i.e., the highest) signals that are revealed after each round. When there are three bidders and two objects, the single-round uniform-price auction is equivalent to an ascending auction. Thus, our revenue ranking results imply that, for some model parameters, a sequential auction with winning-bids announcement yields higher revenue than any standard single-round auction.

While there are many papers on sequential auctions with bidders having independent private values (see Klemperer, 1999, and Krishna, 2002, for surveys), sequential auctions with affiliated values have been little studied. Two papers somewhat connected to our work are Ortega Reichert (1968) and Hausch (1986). In both papers, bidders demand more than one object. Ortega Reichert (1968) studies a two-bidder, two-period, sequential first-price auction with positive correlation of bidders' valuations across periods and across bidders. He shows that there is a deception effect. Compared to a one-shot auction, bidders reduce their first-round bids to induce rivals to hold more pessimistic beliefs about their valuations for the second object. Hausch (1986) studies a special discrete case of a two-bidder, two-unit demand, two-signal, two-period, common-value, sequential first-price auction in which both the losing and the winning bids are announced after the first round. Besides the deception effect, he shows that there is an opposite informational effect that raises the seller's revenue. In our model bidders have unit-demand, so there is no deception effect; with no bid announcement first-round bids are lower because bidders condition on being tied with the price setter, not because they want to deceive their opponents. Furthermore, when the seller reveals the first-round winning bids, there are informational effects on both first and second-round bidding.

We introduce the model in the next section. Section 3 studies the symmetric equilibria of the sequential auction with and without winning-bids announcement. Section 4 contains the proofs of the theorems reported in Section 3 and discusses extensions to more than two rounds. Section 5 compares the price sequences and revenues in the sequential and single-round auctions. Section 6 concludes.

2 The Model

We consider the standard affiliated-value model of Milgrom and Weber (1982, 2000). A seller owns k identical objects. There are n bidders participating in the auction, every bidder desiring only one object. Before the auction, bidder i, $i = 1, 2, \ldots, n$, observes the realization x_i of a signal X_i. Let $s_1, ..., s_m$ be the realizations of additional signals $S_1, ..., S_m$ unobservable by the bidders, and denote with w the vector of signal realizations $(s_1, ..., s_m, x_1, ..., x_n)$. Let $w \vee w'$ be the component-wise maximum and $w \wedge w'$ be the component-wise minimum of w and w'. Signals are drawn from a distribution with a joint pdf $f(w)$, which is symmetric in its last n arguments (the signals x_i) and satisfies the affiliation property:

$$f(w \vee w')f(w \wedge w') \geq f(w)f(w') \qquad \text{for all } w, w'. \tag{1}$$

The support of f is $[\underline{s}, \overline{s}]^m \times [\underline{x}, \overline{x}]^n$, with $-\infty \leq \underline{s} < \overline{s} \leq +\infty$, and $-\infty \leq \underline{x} < \overline{x} \leq +\infty$.

The value of one object for bidder i is given by $V_i = u(S_1, ..., S_m, X_i, \{X_j\}_{j \neq i})$, where the function $u(\cdot)$ satisfies the following assumption.

Assumption 1. $V_i = u(S_1, ..., S_m, X_i, \{X_j\}_{j \neq i})$ *is non-negative, bounded, continuous, in-*

creasing in each variable, and symmetric in the other bidders' signals X_j, $j \neq i$.

We compare two standard auction formats. In a *single-round uniform auction* (an extension of the second-price auction introduced by Vickrey, 1961) the seller auctions all objects simultaneously in a single round. The bidders with the k highest bids win one object each at a price equal to the $(k+1)$-st highest bid. In a *sequential uniform auction*, the seller auctions the objects in two rounds, k_1 objects in the first round and $k_2 = k - k_1$ in the second round. In round t, $t = 1, 2$, the bidders with the k_t highest bids win one object each at a price equal to the $(k_t + 1)$-st highest bid. Since bidders have unit demand, only the $n - k_1$ first-round losers participate in the second round. For the sequential auction, we consider two information policies. According to the first policy, the seller does not reveal any information after the first round. This is referred to as the *no-bid-announcement* policy. The second policy prescribes that the seller announces the first-round winning bids (i.e., the k_1 highest bids), before the second round bids are submitted. This is referred to as the *winning-bids-announcement* policy.

3 Symmetric Equilibria

To derive the symmetric equilibrium bidding functions in each of the auction formats, it is useful to take the point of view of one of the bidders, say bidder 1 with signal $X_1 = x$, and to consider the order statistics associated with the signals of all other bidders. We denote with Y^m the m-th highest signal of bidders $2, 3, ..., n$ (i.e., all bidders except bidder 1).

A very important implication of affiliation is that if H is an increasing function, then $E\left[H\left(X_1, Y^1, ..., Y^k\right)|c_1 \leq Y^1 \leq d_1, ..., c_k \leq Y^k \leq d_k\right]$ is increasing in all its arguments (see Milgrom and Weber, 1982, Theorem 5). We use this property repeatedly in our proofs; when we refer to affiliation, we refer to this property.

We denote with $b^s(\cdot)$ the symmetric equilibrium bidding function of the single-round uniform auction; $b^n_t(\cdot)$ and $b^a_t(\cdot)$ are the symmetric equilibrium bidding functions in round t, $t = 1, 2$, of the sequential uniform auction with no bid announcement and with winning-bids announcement, respectively.

We begin by recalling (see Milgrom and Weber, 1982, 2000) that a symmetric equilibrium bidding function in the single-round uniform auction is:

$$b^s(x) = E\left[V_1 | X_1 = x, Y^k = x\right]. \tag{2}$$

Due to affiliation and Assumption 1, $b^s(x)$ is an increasing function of x. Bidder 1 bids the expected value of an object conditional on his own signal, $X_1 = x$, and on his signal being just high enough to guarantee winning (i.e., being equal to the k-th highest signal among all other bidders' signals). This last conditioning can be understood as a winner's curse correction due to interdependent values. If values are private, bidders need not condition their bids on being tied with the price setter.

Theorem 1. *A symmetric equilibrium strategy in the sequential uniform auction with no bid announcement is given by*

$$b_2^n(x) = E\left[V_1 | X_1 = x, Y^k = x\right], \tag{3}$$

$$b_1^n(x) = E\left[b_2^n\left(Y^k\right) | X_1 = x, Y^{k_1} = x\right]. \tag{4}$$

In an auction with no bid announcement, bids in both rounds depend only on a bidder's own signal. In a sequential auction with winning-bids announcement, the second-round bid must also depend on the first-round winning bids. If the first-round symmetric-equilibrium bidding function is increasing (as shown below), announcing the winning bids is equivalent to announcing the k_1 highest signals. Taking the point of view of a bidder who is bidding in the second round, without loss of generality bidder 1, the announced bids reveal the realizations $y_1, ..., y_{k_1}$ of $Y^1, ..., Y^{k_1}$, the k_1 highest signals among bidders $2, ..., n$.

Theorem 2. *Let $y_1, ..., y_{k_1}$ be the realizations of the signals that correspond to the winning bids in the first round. A symmetric equilibrium strategy in the sequential uniform auction* *with winning-bids announcement is given by*

$$b_2^a(x; y_1, ..., y_{k_1}) = E\left[V_1|X_1 = x, Y^1 = y_1, ..., Y^{k_1} = y_{k_1}, Y^k = x\right], \tag{5}$$

$$b_1^a(x) = E\left[b_2^a\left(Y^k; Y^1, ..., Y^{k_1-1}, x\right)|X_1 = x, Y^{k_1} = x\right]. \tag{6}$$

The proofs are left to Section 4; here we discuss the underlying intuition.

First, note that affiliation and Assumption 1 imply that the bidding functions (3), (4), and (6) are increasing in x, while the bidding function (5) is increasing in x and $y_1, ..., y_{k_1}$.

Second, observe that the bidding function in the second round of the auction with no bid announcement (3) coincides with the bidding function in the single-round uniform auction (2). Intuitively, this makes sense. With no announcement, the only additional information that bidders have in the second round is that in the first round k_1 bidders bid higher than they did. Since the first-round bid function is increasing, this implies that the remaining bidders know that k_1 of the signals of the other bidders are higher than their own. Thus, a bidder bids the expected value of an object conditional on (a) his own signal, (b) the fact that the k_1 first-round winners have higher signals, and (c) his own signal being just high enough to win (i.e., being equal to the $(k - k_1)$-th highest signal of the $n - 1 - k_1$ remaining opponents). This is equivalent to saying that a bidder conditions on his own signal and on his signal being equal to the k-th highest signal of the other $n - 1$ bidders, which yields the same equilibrium bidding function as in a single-round uniform auction.

Third, the second-round bidding function for the case in which the first-round winning bids are announced must also condition on the signals revealed by this announcement. In this case, each remaining bidder bids the expected value of an object conditional on (a) his own signal, (b) his own signal being just high enough to win (i.e., being equal to the k-th highest signal of his opponents), and (c) the revealed signal values of the first-round winning bidders.

Finally, in the first round of a sequential auction with or without winning-bids announcement, a bidder knows that, if he loses, he will get another chance to win the object; hence, he does not want to pay more than what he expects to pay in the second round. He bids the expected second-round price conditional on his own signal and his own signal being just high enough to win in the first round (i.e., being equal to the k_1-th highest signal of the opponents). The second-round price is the second-round bid of the opponent with the k-th highest signal: $b_2^n(Y^k)$ in an auction with no bid announcement and $b_2^a\left(Y^k; Y^1, ..., Y^{k_1}\right)$ in an auction with winning-bids announcement.

4 Proofs of Theorems 1 and 2

Although the proofs are an important contribution of this paper, this section can be skipped by the reader who is only interested in the properties and comparisons of the different auction formats. We first prove Theorem 2. This proof is simpler and allows us to highlight the difficulties in the proof of Theorem 1. We will conclude this section with a remark about extending Theorems 1 and 2 to more than two rounds of bidding.

Proof of Theorem 2. Assume that all bidders other than bidder 1 use the bidding functions $b_1^a(\cdot)$ and $b_2^a(\cdot)$. We need to show that bidder 1 also wants to use them. Suppose, to the contrary, that bidder 1 observes signal x and bids β_1 in the first round. Moreover, if he does not win in the first round, the signals $y_1, ..., y_{k_1}$ corresponding to the k_1 highest bids in the first round are revealed, and bidder 1 bids $\beta_2(y_1, ..., y_{k_1})$ in the second round. Bidder 1 would not profit from bidding outside the range of the bidding functions $b_1^a(\cdot)$ and $b_2^a(\cdot)$; bidding below $\min(b_1^a)$ is equivalent to bidding $\min(b_1^a)$ - in this case bidder 1 never wins - while bidding above $\max(b_1^a)$ is equivalent to bidding $\max(b_1^a)$ - in this case bidder 1 always wins. Since the bidding functions are continuous, we can define σ_1 and $\sigma_2(y_1, ..., y_{k_1})$ such that $b_1^a(\sigma_1) = \beta_1$ and $b_2^a\big(\sigma_2(y_1, ..., y_{k_1}), y_1, ..., y_{k_1}\big) = \beta_2(y_1, ..., y_{k_1})$; that is, we can think that bidder 1 uses the same bidding functions as all other bidders, but in the first round he bids as if he had observed signal σ_1, and in the second round as if he had observed signal $\sigma_2(\cdot)$.

Define

$$v_2(x; y_1, ..., y_{k_1}; y_k) = E\left[V_1 | X_1 = x, Y^1 = y_1, Y^2 = y_2, ..., Y^{k_1} = y_{k_1}, Y^k = y_k\right], \qquad (7)$$

and note from (5) that

$$b_2^a(x; y_1, ..., y_{k_1}) = v_2(x; y_1, ..., y_{k_1}; x). \tag{8}$$

Suppose that bidder 1 does not win an object in the first-round. Then, using (8), if he bids as if his signal were σ_2 in the second round, his expected profit conditional on $X_1 = x$ and $Y^1 = y_1, ..., Y^{k_1} = y_{k_1}$ is

$$\begin{aligned} \pi_2^a(x; y_1, ..., y_{k_1}; \sigma_2) &= \int_{\underline{x}}^{\sigma_2} \Big(v_2(x; y_1, .., y_{k_1}; y_k) - b_2^a(y_k; y_1, .., y_{k_1}) \Big) h(y_k | x; y_1, .., y_{k_1}) dy_k \quad (9) \\ &= \int_{\underline{x}}^{\sigma_2} \Big(v_2(x; y_1, .., y_{k_1}; y_k) - v_2(y_k; y_1, .., y_{k_1}; y_k) \Big) h(y_k | x; y_1, .., y_{k_1}) dy_k, \end{aligned}$$

where $h(y_k|x; y_1, .., y_{k_1})$ is the density of Y^k conditional on $X_1 = x$, $Y^1 = y_1, ..., Y^{k_1} = y_{k_1}$. Affiliation and Assumption 1 imply that $v_2(x; y_1, ..., y_{k_1}; y_k)$ is increasing in x, so the difference $v_2(x; y_1, ..., y_{k_1}; y_k) - v_2(y_k; y_1, ..., y_{k_1}; y_k)$ has the same sign as $x - y_k$. Hence bidder 1's expected second-round profit is maximized by setting $\sigma_2 = x$, and bidder 1's optimal bid in the second round is $b_2^a(x; y_1, ..., y_{k_1})$. This establishes that, no matter what he bid in the first round, it is optimal for bidder 1 to bid according to the equilibrium bidding function $b_2^a(\cdot)$ in the second round.

Next, we show that it is also optimal to bid according to $b_1^a(\cdot)$ in the first round. We need some additional notation; let

$$\begin{aligned} v_1(x; y_{k_1}; y_k) &= E\left[V_1 | X_1 = x, Y^{k_1} = y_{k_1}, Y^k = y_k\right] \quad (10) \\ &= E\left[v_2(x; Y^1, ..., Y^{k_1 - 1}, y_{k_1}; y_k) | X_1 = x, Y^{k_1} = y_{k_1}, Y^k = y_k\right], \end{aligned}$$

$$b_2^*(y_k; y_{k_1}|x) = E\left[b_2^a(y_k; Y^1, ..., Y^{k_1 - 1}, y_{k_1}) | X_1 = x, Y^{k_1} = y_{k_1}, Y^k = y_k\right], \tag{11}$$

where the second equality in (10) follows because, by (7), it is

$$\begin{aligned} &E\left[v_2(x; Y^1, ..., Y^{k_1 - 1}, y_{k_1}; y_k) | X_1 = x, Y^{k_1} = y_{k_1}, Y^k = y_k\right] = \\ &\quad E\left[E\left[V_1 | X_1 = x, Y^1, ..., Y^{k_1 - 1}, Y^{k_1} = y_{k_1}, Y^k = y_k\right] | X_1 = x, Y^{k_1} = y_{k_1}, Y^k = y_k\right]. \end{aligned}$$

Bidder 1's total expected profit at the beginning of the first round can be decomposed in two parts: the expected profit from the first round and the expected profit from the second round. Using (9), (10), (11), and recalling that in the first round bidder 1 bids as if his signal were σ_1 and in the second round he bids according to his true signal x, the expected profit from the second round can be written as

$$\begin{aligned}
&\int_{\sigma_1}^{\overline{x}} \dots \int_{\sigma_1}^{\overline{x}} \pi_2^a\left(x; y_1, .., y_{k_1}; x\right) h(y_1, .., y_{k_1}|x) dy_{k_1} \dots dy_1 \\
&= \int_{\sigma_1}^{\overline{x}} \dots \int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{x} \Big(v_2(x; y_1, .., y_{k_1}; y_k) - b_2^a(y_k; y_1, .., y_{k_1}) \Big) h(y_k|x; y_1, .., y_{k_1}) h(y_1, .., y_{k_1}|x) dy_k dy_{k_1} \dots dy_1 \\
&= \int_{\sigma_1}^{\overline{x}} \dots \int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{x} \Big(v_2(x; y_1, .., y_{k_1}; y_k) - b_2^a(y_k; y_1, .., y_{k_1}) \Big) h(y_1, .., y_{k_1}, y_k|x) dy_k dy_{k_1} \dots dy_1 \\
&= \int_{\underline{x}}^{x} \int_{\sigma_1}^{\overline{x}} \dots \int_{\sigma_1}^{\overline{x}} \Big(v_2(x; y_1, .., y_{k_1}; y_k) - b_2^a(y_k; y_1, .., y_{k_1}) \Big) h(y_1, .., y_{k_1-1}|x; y_{k_1}, y_k) h(y_{k_1}, y_k|x) dy_k dy_{k_1} \dots dy_1 \\
&= \int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{x} \Big(v_1(x; y_{k_1}; y_k) - b_2^*(y_k; y_{k_1}|x) \Big) h(y_{k_1}, y_k|x) dy_k dy_{k_1},
\end{aligned}$$

where $h(y_1, ..., y_{k_1}|x)$ is the joint density of $Y^1, ..., Y^{k_1}$ conditional on $X_1 = x$; $h(y_1, ..., y_{k_1}, y_k|x)$ is the joint density of $Y^1, ..., Y^{k_1}, Y^k$ conditional on $X_1 = x$; $h(y_1, .., y_{k_1-1}|x; y_{k_1}, y_k)$ is the joint density of $Y^1, ..., Y^{k_1-1}$ conditional on $X_1 = x$, $Y^{k_1} = y_{k_1}$, and $Y^k = y_k$; $h(y_{k_1}, y_k|x)$ is the joint density of Y^{k_1} and Y^k conditional on $X_1 = x$. Hence, bidder 1's total expected profit at the beginning of the first round is

$$\begin{aligned}
\Pi^a(x; \sigma_1) = &\int_{\underline{x}}^{\sigma_1} \int_{\underline{x}}^{y_{k_1}} \Big(v_1(x; y_{k_1}; y_k) - b_1^a(y_{k_1}) \Big) h(y_{k_1}, y_k|x) dy_k dy_{k_1} \\
&+ \int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{x} \Big(v_1(x; y_{k_1}; y_k) - b_2^*(y_k; y_{k_1}|x) \Big) h(y_{k_1}, y_k|x) dy_k dy_{k_1},
\end{aligned}$$

where the first term is the profit from the first round. Differentiating $\Pi^a(x; \sigma_1)$ with respect to σ_1 yields

$$\begin{aligned}
\frac{\partial \Pi^a(x;\sigma_1)}{\partial \sigma_1} &= \int_{\underline{x}}^{\sigma_1} \Big(v_1(x;\sigma_1;y_k) - b_1^a(\sigma_1)\Big) h(\sigma_1, y_k|x) dy_k \\
&\quad - \int_{\underline{x}}^{x} \Big(v_1(x;\sigma_1;y_k) - b_2^*(y_k;\sigma_1|x)\Big) h(\sigma_1, y_k|x) dy_k \\
&= \int_{\underline{x}}^{\sigma_1} \Big(b_2^*(y_k;\sigma_1|x) - b_1^a(\sigma_1)\Big) h(\sigma_1, y_k|x) dy_k \\
&\quad + \int_{x}^{\sigma_1} \Big(v_1(x;\sigma_1;y_k) - b_2^*(y_k;\sigma_1|x)\Big) h(\sigma_1, y_k|x) dy_k.
\end{aligned} \tag{12}$$

To prove that $\Pi^a(x;\sigma_1)$ is maximized at $\sigma_1 = x$, we will show that $\frac{\partial \Pi^a(x;\sigma_1)}{\partial \sigma_1}$ has the same sign as $(x - \sigma_1)$. The second term in (12) is zero for $\sigma_1 < x$ (by definition, $Y^{k_1} \geq Y^k$ and hence $h(\sigma_1, y_k|x) = 0$ for $\sigma_1 < x$), while for $\sigma_1 > x$ it is negative because

$$\begin{aligned}
& \int_{x}^{\sigma_1} \Big(v_1(x;\sigma_1;y_k) - b_2^*(y_k;\sigma_1|x)\Big) h(\sigma_1, y_k|x) dy_k \\
= & \int_{x}^{\sigma_1} E\left[v_2\left(x;Y^1,...,Y^{k_1-1},\sigma_1;y_k\right)|X_1 = x, Y^{k_1} = \sigma_1, Y^k = y_k\right] h(\sigma_1, y_k|x) dy_k \\
& - \int_{x}^{\sigma_1} E\left[b_2^a\left(y_k;Y^1,...,Y^{k_1-1},\sigma_1\right)|X_1 = x, Y^{k_1} = \sigma_1, Y^k = y_k\right] h(\sigma_1, y_k|x) dy_k \\
= & \int_{x}^{\sigma_1} E\left[v_2\left(x;Y^1,...,Y^{k_1-1},\sigma_1;y_k\right)|X_1 = x, Y^{k_1} = \sigma_1, Y^k = y_k\right] h(\sigma_1, y_k|x) dy_k \\
& - \int_{x}^{\sigma_1} E\left[v_2\left(y_k;Y^1,...,Y^{k_1-1},\sigma_1;y_k\right)|X_1 = x, Y^{k_1} = \sigma_1, Y^k = y_k\right] h(\sigma_1, y_k|x) dy_k \leq 0,
\end{aligned}$$

where the first equality follows from (10) and (11), the second equality follows from (8), and the inequality follows from affiliation and $y_k \geq x$.

By (11) and (6), the first term in (12) is equal to

$$\begin{aligned}
& \int_{\underline{x}}^{\sigma_1} E\left[b_2^a\left(y_k;Y^1,...,Y^{k_1-1},\sigma_1\right)|X_1 = x, Y^{k_1} = \sigma_1, Y^k = y_k\right] h(\sigma_1, y_k|x) dy_k \\
& - b_1^a(\sigma_1) \int_{\underline{x}}^{\sigma_1} h(\sigma_1, y_k|x) dy_k \\
= \; & E\left[b_2^a\left(Y^k;Y^1,...,Y^{k_1-1},\sigma_1\right)|X_1 = x, Y^{k_1} = \sigma_1\right] \int_{\underline{x}}^{\sigma_1} h(\sigma_1, y_k|x) dy_k \\
& - E\left[b_2^a\left(Y^k;Y^1,...,Y^{k_1-1},\sigma_1\right)|X_1 = \sigma_1, Y^{k_1} = \sigma_1\right] \int_{\underline{x}}^{\sigma_1} h(\sigma_1, y_k|x) dy_k.
\end{aligned}$$

Because of affiliation, this difference has the same sign as $(x - \sigma_1)$. Hence $\frac{\partial \Pi^a(x;\sigma_1)}{\partial \sigma_1}$ is positive for $\sigma_1 < x$ and negative for $\sigma_1 > x$. Therefore, the expected profit of bidder 1 is maximized at $\sigma_1 = x$. This implies that bidder 1's optimal bid in the first round is $b_1^a(x)$ and concludes the proof. ∎

There are two steps in the proof of Theorem 2. Assuming that all other bidders follow the bidding functions $b_1^a(\cdot)$ and $b_2^a(\cdot)$, first we show that, no matter what bidder 1 did in the first round, in the second round it is optimal for him to bid according to $b_2^a(\cdot)$. Then we show that in the first round it is optimal to follow $b_1^a(\cdot)$. This method of proof does not fully generalize to the case of no bid announcement. In this case, it is optimal for bidder 1 with signal x to bid according to $b_2^n(x)$ in the second round if and only if he has bid according to $b_1^n(x)$, or lower, in the first round. On the contrary, if bidder 1 has bid higher than $b_1^n(x)$ in the first round and lost, he will want to bid higher than $b_2^n(x)$ in the second round. This, in turn, makes it difficult to show that it is optimal to bid according to $b_1^n(x)$ in the first round. As Milgrom and Weber (2000, p. 182) point out, the difficulty in proving equilibrium existence in this case is in ruling out that "a bidder might choose to bid a bit higher in the first round in order to have a better estimate of the winning bid, should he lose." Our proof of Theorem 1 overcomes this difficulty by using the following lemma, proven in Appendix A.

Lemma 1. *Let $D(s)$ be a continuous function defined on $[0, S]$. Let $a(s)$ be a non-decreasing positive function, defined on $[0, S]$. If $\int_0^x D(s)a(s)ds \leq 0$ for all $x \in [0, S]$, then $\int_0^z D(s)ds \leq 0$ for all $z \in [0, S]$.*

Proof of Theorem 1. We begin as in the proof of Theorem 2, by assuming that all bidders other than bidder 1 use the bidding functions $b_1^n(\cdot)$ and $b_2^n(\cdot)$ given by (4) and (3). We want to show that it is also optimal for bidder 1 to use them. Suppose, to the contrary, that bidder 1 observes signal x and bids β_1 in the first round and β_2 in the second round. As in the proof of Theorem 2, we can define σ_1 and σ_2 such that $b_1^n(\sigma_1) = \beta_1$ and $b_2^n(\sigma_2) = \beta_2$.

We begin by showing that if bidder 1 bids less than or equal to $b_1^n(x)$ in the first round (i.e., $\sigma_1 \leq x$), then his optimal bid in the second round is $b_2^n(x)$ (i.e., $\sigma_2 = x$ maximizes his expected second round profit).

If bidder 1 does not win an object in the first round, he knows that $y_{k_1} > \sigma_1$. Then his expected second-round profit conditional on σ_1, σ_2, and $X_1 = x$ can be written as

$$\pi_2^n(x;\sigma_1,\sigma_2) = \int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{\sigma_2} \Big(v_1(x;y_{k_1};y_k) - b_2^n(y_k) \Big) \frac{h(y_{k_1}, y_k | x)}{\int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{\widetilde{y}_{k_1}} h(\widetilde{y}_{k_1}, \widetilde{y}_k | x) d\widetilde{y}_k d\widetilde{y}_{k_1}} dy_k dy_{k_1}, \quad (13)$$

where $v_1(\cdot)$ is given by (10) and $h(y_{k_1}, y_k|x)$ is the joint density of Y^{k_1} and Y^k conditional on $X_1 = x$.[2] Differentiating $\pi_2^n(x;\sigma_1,\sigma_2)$ with respect to σ_2, we obtain

$$\frac{\partial \pi_2^n(x;\sigma_1,\sigma_2)}{\partial \sigma_2} = \int_{\sigma_1}^{\overline{x}} \Big(v_1(x;y_{k_1};\sigma_2) - b_2^n(\sigma_2) \Big) \frac{h(y_{k_1}, \sigma_2 | x)}{\int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{\widetilde{y}_{k_1}} h(\widetilde{y}_{k_1}, \widetilde{y}_k | x) d\widetilde{y}_k d\widetilde{y}_{k_1}} dy_{k_1},$$

which is equal to

$$\big(E[V_1 | X_1 = x, Y^{k_1} \geq \sigma_1, Y^k = \sigma_2] - E[V_1 | X_1 = \sigma_2, Y^k = \sigma_2] \big) \frac{\int_{\underline{x}}^{\widetilde{y}_{k_1}} h(\widetilde{y}_{k_1}, \sigma_2 | x) d\widetilde{y}_{k_1}}{\int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{\widetilde{y}_{k_1}} h(\widetilde{y}_{k_1}, \widetilde{y}_k | x) d\widetilde{y}_k d\widetilde{y}_{k_1}}.$$

We now show that, when $\sigma_1 \leq x$, $\frac{\partial \pi_2^n(x;\sigma_1,\sigma_2)}{\partial \sigma_2}$ has the same sign as $(x - \sigma_2)$.

First, consider $\sigma_2 > x$. Then

$$E[V_1 | X_1 = x, Y^{k_1} \geq \sigma_1, Y^k = \sigma_2] = E[V_1 | X_1 = x, Y^k = \sigma_2] \leq E[V_1 | X_1 = \sigma_2, Y^k = \sigma_2],$$

where the equality follows from $\sigma_2 > x \geq \sigma_1$ and $Y^{k_1} \geq Y^k$, and the inequality follows from affiliation and Assumption 1. We conclude that $\frac{\partial \pi_2^n(x;\sigma_1,\sigma_2)}{\partial \sigma_2}$ is negative for $\sigma_2 > x$.

Second, consider $\sigma_2 < x$. Then, by affiliation and Assumption 1,

$$E[V_1 | X_1 = x, Y^{k_1} \geq \sigma_1, Y^k = \sigma_2] \geq E[V_1 | X_1 = x, Y^k = \sigma_2] \geq E[V_1 | X_1 = \sigma_2, Y^k = \sigma_2],$$

so $\frac{\partial \pi_2^n(x;\sigma_1,\sigma_2)}{\partial \sigma_2}$ is positive in this case. Thus, if bidder 1 bids less than or equal to $b_1^n(x)$ in the first round ($\sigma_1 \leq x$), his optimal bid in the second round is $b_2^n(x)$.

We complete the proof by showing that it is optimal for bidder 1 to bid $b_1^n(x)$ in the first

[2] Note that the second-round profit depends on σ_1, contrary to the case in which the winning bids are announced, see equation (9).

round (i.e., to bid as if $\sigma_1 = x$). Let $\sigma_2^*(\sigma_1)$ be the value of σ_2 that maximizes $\pi_2^n(x;\sigma_1,\sigma_2)$; we have already shown that $\sigma_2^*(\sigma_1) = x$ for $\sigma_1 \leq x$. Using (10), bidder 1's total expected profit at the beginning of the first round is

$$\begin{aligned}
\Pi^n(x;\sigma_1) &= \int_{\underline{x}}^{\sigma_1}\int_{\underline{x}}^{y_{k_1}} \Big(v_1(x;y_{k_1};y_k) - b_1^n(y_{k_1})\Big) h(y_{k_1}, y_k|x) dy_k dy_{k_1} \\
&\quad + \int_{\sigma_1}^{\overline{x}}\int_{\underline{x}}^{\sigma_2^*(\sigma_1)} \Big(v_1(x;y_{k_1};y_k) - b_2^n(y_k)\Big) h(y_{k_1}, y_k|x) dy_k dy_{k_1},
\end{aligned}$$

where the second term follows from (13). Differentiating with respect to σ_1 and applying the envelope theorem, we obtain

$$\begin{aligned}
\frac{\partial \Pi^n(x;\sigma_1)}{\partial \sigma_1} &= \int_{\underline{x}}^{\sigma_1} \Big(v_1(x;\sigma_1;y_k) - b_1^n(\sigma_1)\Big) h(\sigma_1, y_k|x) dy_k \\
&\quad - \int_{\underline{x}}^{\sigma_2^*(\sigma_1)} \Big(v_1(x;\sigma_1;y_k) - b_2^n(y_k)\Big) h(\sigma_1, y_k|x) dy_k \\
&= \int_{\underline{x}}^{\sigma_1} \Big(b_2^n(y_k) - b_1^n(\sigma_1)\Big) h(\sigma_1, y_k|x) dy_k \qquad (14) \\
&\quad + \int_{\sigma_2^*(\sigma_1)}^{\sigma_1} \Big(v_1(x;\sigma_1;y_k) - b_2^n(y_k)\Big) h(\sigma_1, y_k|x) dy_k.
\end{aligned}$$

The first term in equation (14), using (4), is

$$\begin{aligned}
&\left(\int_{\underline{x}}^{\sigma_1} \Big(b_2^n(y_k) - b_1^n(\sigma_1)\Big) \frac{h(\sigma_1, y_k|x)}{\int_{\underline{x}}^{\sigma_1} h(\sigma_1, \widetilde{y}_k|x) d\widetilde{y}_k} dy_k\right) \int_{\underline{x}}^{\sigma_1} h(\sigma_1, \widetilde{y}_k|x) d\widetilde{y}_k = \\
&\quad \Big(E\big[b_2^n(Y^k)\big|X_1 = x, Y^{k_1} = \sigma_1\big] - E\big[b_2^n(Y^k)\big|X_1 = \sigma_1, Y^{k_1} = \sigma_1\big]\Big) \int_{\underline{x}}^{\sigma_1} h(\sigma_1, y_k|x) dy_k.
\end{aligned}$$

By affiliation, this term has the same sign as $(x - \sigma_1)$.

We now show that the second term in (14) is non-negative for $\sigma_1 < x$ and non-positive for $\sigma_1 > x$. As a result, $\frac{\partial \Pi^n(x;\sigma_1)}{\partial \sigma_1}$ has the same sign as $(x - \sigma_1)$, which implies that $\sigma_1 = x$ maximizes $\Pi^n(x;\sigma_1)$; that is, the optimal bid of bidder 1 in the first round is $b_1^n(x)$. This will conclude the proof. First, observe that if $\sigma_2^*(\sigma_1) \geq \sigma_1$ then the second term in (14) is zero, since, by definition, $h(\sigma_1, y_k|x) = 0$ for $y_k > \sigma_1$. Therefore, since $\sigma_2^*(\sigma_1) = x$ for $\sigma_1 \leq x$, the second term in (14) is zero for $\sigma_1 \leq x$.

It remains to show that in the case $\sigma_1 > x$ and $\sigma_2^*(\sigma_1) < \sigma_1$ the second term in (14) is non-positive.[3] We will use Lemma 1. Define

$$D(y_k) = \left(\frac{\int_{\sigma_1}^{\overline{x}} v_1(x; y_{k_1}; y_k) h(y_{k_1}, y_k|x) dy_{k_1}}{\int_{\sigma_1}^{\overline{x}} h(y_{k_1}, y_k|x) dy_{k_1}} - b_2^n(y_k) \right) h(\sigma_1, y_k|x), \tag{15}$$

and note that, by affiliation, the second term in (14) is smaller than $\int_{\sigma_2^*(\sigma_1)}^{\sigma_1} D(y_k) dy_k$. Then a sufficient condition for the second term in (14) to be negative is

$$\int_{\sigma_2^*(\sigma_1)}^{\sigma_1} D(y_k) dy_k \leq 0. \tag{16}$$

Define

$$a(y_k) = \frac{\int_{\sigma_1}^{\overline{x}} h(y_{k_1}, y_k|x) dy_{k_1}}{h(\sigma_1, y_k|x)}. \tag{17}$$

Since $\sigma_2^*(\sigma_1)$ maximizes $\pi_2^n(x; \sigma_1, \sigma_2)$, by (13) we have, for all σ_2,

$$\int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{\sigma_2^*(\sigma_1)} \Big(v_1(x; y_{k_1}; y_k) - b_2^n(y_k) \Big) h(y_{k_1}, y_k|x) dy_k dy_{k_1} \geq$$

$$\int_{\sigma_1}^{\overline{x}} \int_{\underline{x}}^{\sigma_2} \Big(v_1(x; y_{k_1}; y_k) - b_2^n(y_k) \Big) h(y_{k_1}, y_k|x) dy_k dy_{k_1}.$$

Rearranging terms yields, for all σ_2,

$$\int_{\sigma_2^*(\sigma_1)}^{\sigma_2} \int_{\sigma_1}^{\overline{x}} \Big(v_1(x; y_{k_1}; y_k) - b_2^n(y_k) \Big) h(y_{k_1}, y_k|x) dy_{k_1} dy_k \leq 0. \tag{18}$$

Using the definitions (15) and (17) of $D(y_k)$ and $a(y_k)$, expression (18) can be rewritten as

$$\int_{\sigma_2^*(\sigma_1)}^{\sigma_2} \left(\frac{\int_{\sigma_1}^{\overline{x}} v_1(x; y_{k_1}; y_k) h(y_{k_1}, y_k|x) dy_{k_1}}{\int_{\sigma_1}^{\overline{x}} h(y_{k_1}, y_k|x) dy_{k_1}} - b_2^n(y_k) \right) \left(\int_{\sigma_1}^{\overline{x}} h(y_{k_1}, y_k|x) dy_{k_1} \right) dy_k$$

$$= \int_{\sigma_2^*(\sigma_1)}^{\sigma_2} D(y_k) a(y_k) dy_k \leq 0 \qquad \text{for all } \sigma_2. \tag{19}$$

[3]This is precisely where lies the difficulty mentioned by Milgrom and Weber (2000). We need to show that in the first round bidder 1 does not want to bid as if his signal were higher than x.

Note that $a(y_k)$, given by (17), is positive and increasing (by affiliation). Then, by Lemma 1, equation (19) implies that $\int_{\sigma_2^*(\sigma_1)}^{z} D(y_k)dy_k \leq 0$ for all $z \geq \sigma_2^*(\sigma_1)$; in particular, $\int_{\sigma_2^*(\sigma_1)}^{\sigma_1} D(y_k)dy_k \leq 0$, so (16) holds. Thus, the second term in (14) is negative, which concludes the proof. ▮

REMARK. Theorem 2 and its proof readily generalize to the case of any finite number of rounds. Suppose that there are T rounds of bidding and k_t objects are sold in round t. Let $m_t = \sum_{\tau=1}^{t} k_\tau$. Then, in the last round the symmetric equilibrium bidding function of the sequential auction with winning-bids announcement is

$$b_T^a(x; y_1, ..., y_{m_{T-1}}) = E\left[V_1 | X_1 = x, Y^1 = y_1, ..., Y^{m_{T-1}} = y_{m_{T-1}}, Y^{m_T} = x,\right],$$

while for $t < T$ the bidding functions are

$$b_t^a(x; y_1, ..., y_{m_{t-1}}) =$$
$$= E\left[b_{t+1}^a\left(Y^{m_{t+1}}; y_1, ..., y_{m_{t-1}}, Y^{m_{t-1}+1}, ..., Y^{m_t-1}, x\right) | X_1 = Y^{m_t} = x, Y^1 = y_1, ..., Y^{m_{t-1}} = y_{m_{t-1}}\right].$$

In the case of no bid announcement, this extension presents a technical difficulty, as it is not clear how to generalize the proof of Theorem 1.

5 Properties of Sequential Auctions

This section establishes the properties and compares the equilibrium bidding strategies of the single-round and the sequential uniform auctions with and without winning-bids announcement. It is now convenient to take the point of view of the seller, or of an outside observer, and consider the order statistics of the signals of all n bidders. Denote with Z^m the m-th highest signal among all n bidders. It is important to observe that, because of the symmetry of signals, conditioning on the event $\{Z^{m+1} = x\}$ is equivalent to conditioning on the event $\{X_1 \geq x, Y^m = x\}$, or on the event $\{Y^m \geq X_1 = x \geq Y^{m+1}\}$. The event that the $(m+1)$-st highest signal is x is equivalent to the event that one bidder, without loss of generality bidder 1, has a signal higher than or equal to x, and the m-th highest signal among all other bidders' signals is x. It is also equivalent to the event that bidder 1 has signal x and the m-th highest signal among all other bidders is greater than x, while the $(m+1)$-st highest signal is smaller than x.

We first look at the price sequences. Let P_t^a and P_t^n be the price in round t of a sequential auction with and without winning-bids announcement. The price in each round is a random variable: $P_1^a = b_1^a(Z^{k_1+1})$, $P_2^a = b_2^a(Z^{k+1}; Z^1, ..., Z^{k_1})$, $P_1^n = b_1^n(Z^{k_1+1})$, and $P_2^n = b_2^n(Z^{k+1})$. With winning-bids announcement, conditional on knowing the realization p_1^a of P_1^a, the expected price in the second round is higher than p_1^a; similarly, with no bid announcement, conditional on knowing the realization p_1^n of P_1^n, the expected second-round price is higher than p_1^n. Prices drift upward. Milgrom and Weber (2000) were the first to report this result for their conjectured equilibrium with no bid announcement and one object sold in each round. Here we extend the result to the case of winning-bids announcement.[4]

Theorem 3. *In a sequential uniform auction with or without winning-bids announcement, the expected second-round price conditional on the realized first-round price is higher than the realized first-round price:* $E\left[P_2^a|p_1^a\right] \geq p_1^a$ *and* $E\left[P_2^n|p_1^n\right] \geq p_1^n$.

Proof. Consider the case of winning-bids announcement. By Theorem 2, the realized price in the first round is given by $p_1^a = b_1^a(z_{k_1+1})$, where z_{k_1+1} is the realized value of Z^{k_1+1}, the (k_1+1)-st highest out of n signals. Thus, conditioning on p_1^a is the same as conditioning on $Z^{k_1+1} = z_{k_1+1}$. The price in the second round is $P_2^a = b_2^a(Z^{k+1}; Z^1, ..., Z^{k_1})$. Since conditioning on the event $\{Z^{k_1+1} = z_{k_1+1}\}$ is equivalent to conditioning on the event $\{Y^{k_1} \geq X_1 = z_{k_1+1} \geq Y^{k_1+1}\}$, the expected price in the second round conditional on p_1^a is

$$\begin{aligned} E[P_2^a|p_1^a] &= E[b_2^a(Z^{k+1}; Z^1, ..., Z^{k_1})|Z^{k+1} = z_{k_1+1}] \\ &= E[b_2^a(Y^k; Y^1, ..., Y^{k_1})|Y^{k_1} \geq X_1 = z_{k_1+1} \geq Y^{k_1+1}] \\ &\geq E[b_2^a(Y^k; Y^1, ..., Y^{k_1-1}, z_{k_1+1})|Y^{k_1} = X_1 = z_{k_1+1}] = b_1^a(z_{k_1+1}) = p_1^a, \end{aligned}$$

where the inequality follows from affiliation.

[4]The result easily extends to the case of multiple rounds.

The proof of the no bid announcement case is analogous. Letting $p_1^n = b_1^n(z_{k_1+1})$, we have

$$\begin{aligned} E[P_2^n|p_1^n] &= E[b_2^n(Z^{k+1})|Z^{k_1+1} = z_{k_1+1}] = E[b_2^n(Y^k)|Y^{k_1} \geq X_1 = z_{k_1+1} \geq Y^{k_1+1}] \\ &\geq E[b_2^n(Y^k)|Y^{k_1} = X_1 = z_{k_1+1}] = b_1^n(z_{k_1+1}) = p_1^n. \end{aligned}$$

∎

Theorem 3 has an important implication. The expected, unconditional, price (i.e., the price expected by the seller) in the second round is higher than the expected, unconditional, price in the first round. Therefore, in the single-round uniform auction the seller's expected revenue is always higher than in the sequential uniform auction with no bid announcement.

Theorem 4. *The seller's expected revenue in a sequential uniform auction with no bid announcement is lower than in a single-round uniform auction for k objects.*[5]

Proof. The second round bidding function $b_2^n(x)$, given by (3), is the same as the bidding function in a single-round auction $b^s(x)$, given by (2). Therefore, the expected second-round price in the sequential auction is the same as the expected price in the single-round auction. By Theorem 3, the expected price in the first round of a sequential auction is lower than the expected price in the second round. Thus expected revenue in the sequential auction is lower. ∎

The second-round bid function in a sequential auction with no bid announcement and the bid function in the single-round auction coincide. Thus, auctioning objects in sequence yields no gain in second-round revenue when the winning bids are not announced. Auctioning objects in sequence, however, has a cost to the seller, because it induces bidders to lowball in the first round. Let $P^s = b^s(Z^{k+1})$ be the price in a single-round auction for k objects, and define the *lowballing effect* on revenue, L, as the difference between the expected price in the first round of a sequential auction with no bid announcement and the expected price in a single-round auction:

$$L = E[P_1^n] - E[P^s].$$

[5] Surprisingly, even though they noticed that the price sequence is upward drifting, Milgrom and Weber (2000, p. 193) conjectured that the sequential auction with no bid announcement yields greater revenue than the single-round uniform auction.

By Theorem 3, L is negative. To better understand the lowballing effect, suppose z_{k_1+1} is the signal realization of the price setter in the first round of a sequential auction with no bid announcement. Equilibrium requires that, conditional on being tied with (i.e., having the same signal as) the first-round price setter, a bidder (say bidder 1) is indifferent between winning in the first or in the second round. Formally,

$$\begin{aligned} E\left[P_1^n | X_1 = Y^{k_1} = z_{k_1+1}\right] &= b_1^n(z_{k_1+1}) \\ &= E\left[b_2^n(Y^k) | X_1 = Y^{k_1} = z_{k_1+1}\right] \\ &= E\left[P_2^n | X_1 = Y^{k_1} = z_{k_1+1}\right]. \end{aligned}$$

However, since ties have zero probability, the first-round price setter pays a lower price than the average price he would pay if he won in the second round. Formally, since $P_1^n = b_1^n(Z_{k_1+1})$,

$$\begin{aligned} E\left[P_1^n | X_1 \geq Y^{k_1} = z_{k_1+1}\right] &= E\left[P_1^n | X_1 = Y^{k_1} = z_{k_1+1}\right] \\ &= E\left[P_2^n | X_1 = Y^{k_1} = z_{k_1+1}\right] \\ &\leq E\left[P_2^n | X_1 \geq Y^{k_1} = z_{k_1+1}\right]. \end{aligned}$$

Affiliation is crucial for this result. With private values (e.g., a bidder's value for one object coincides with his signal) and independent signals, the price sequence is a martingale and revenue in the single-round and the sequential uniform auctions coincide (see Milgrom and Weber, 2000, Weber, 1983). On the other hand, if values are private but affiliated, then the price sequence is increasing.

In the second round of a sequential auction with affiliated private values, a bidder bids his own value; that is, the second-round bid coincides with the bid in a single-round uniform auction, irrespective of whether the first-round winning bids are revealed. The first-round bidding functions also do not depend on the information policy. This has the important implication that with affiliated private values auctioning the objects in a single round yields the seller higher revenue independently of the information policy that he follows.

Theorem 5. *With affiliated private values, the seller's expected revenue in a sequential uniform auction with winning-bids announcement is the same as in a sequential uniform auction with no bid announcement and is lower than in a single-round uniform auction for k objects.*

The next result shows that, in the general affiliated model, when the first-round winning bids are announced, there is a positive informational effect on second-round bids.

Theorem 6. *In the sequential auction with winning-bids announcement the expected second-round price is higher than the expected price in a single-round auction for k objects.*

Proof. Let z_{k+1} be the realization of the $(k+1)$-st highest signal among all n bidders. Conditional on $Z^{k+1} = z_{k+1}$, the expected price in a single-round uniform auction for k objects is

$$E[P^s|Z^{k+1} = z_{k+1}] = b^s(z_{k+1}) = E[V_1|X_1 = Y^k = z_{k+1}].$$

Conditioning on $\{Z^{k+1} = z_{k+1}\}$ is equivalent to conditioning on $\{X_1 \geq Y^k = z_{k+1}\}$. Thus, the expected price in the second round of the sequential auction with winning-bids announcement, conditional on $Z^{k+1} = z_{k+1}$, is

$$\begin{aligned} E[P_2^a|Z^{k+1} = z_{k+1}] &= E[b_2^a(z_{k+1}; Y^1, ..., Y^{k_1})|X_1 \geq Y^k = z_{k+1}] \\ &\geq E[b_2^a(z_{k+1}; Y^1, ..., Y^{k_1})|X_1 = Y^k = z_{k+1}] \\ &= E[E[V_1|X_1 = Y^k = z_{k+1}, Y^1, ..., Y^{k_1}]|X_1 = Y^k = z_{k+1}] \\ &= E[V_1|X_1 = Y^k = z_{k+1}] = E[P^s|Z^{k+1} = z_{k+1}], \end{aligned}$$

where the inequality follows from affiliation. Taking expectations over Z^{k+1} concludes the proof. ▮

Define the *second-round informational effect* on revenue, I_2^a, as the difference between the expected prices in the second round of the sequential auctions with and without winning-bids announcement:

$$I_2^a = E\left[P_2^a\right] - E\left[P_2^n\right] = E\left[P_2^a\right] - E\left[P^s\right].$$

By Theorem 6, this informational effect is positive (it is zero in the affiliated private values model) and can be understood in light of the linkage principle first analyzed by Milgrom and Weber (1982): public revelation of a random variable affiliated with bidders's signals raises expected bids and the seller's expected revenue in a single-round auction. Here the revealed random variables are the winning bids in the first round (i.e., the k_1 highest signals among all bidders). Such a revelation raises the expected second-round price. A caveat is in order, however. The fact that the first-round winning bids will be revealed also has an effect on first-round bids. Define the *first-round informational effect* on revenue, I_1^a, as the difference between the expected prices in the first round of the sequential auction with and without winning-bids announcement

$$I_1^a = E\left[P_1^a\right] - E\left[P_1^n\right].$$

Then, the difference between the expected price in the first round of a sequential auction with winning-bids announcement and the expected price in a single-round auction can be viewed as the sum of the lowballing effect and the first-round informational effect

$$E\left[P_1^a\right] - E\left[P^s\right] = L + I_1^a.$$

The policy of revealing the winning bids has a complicated effect on first-round bidding. As demonstrated by the numerical example below, the first-round informational effect can be either positive or negative (by Theorem 5, $I_1^a = 0$ in the affiliated private values model). When it is positive, I_1^a could be greater or smaller than $-L$. As a result, all rankings of the expected first-round prices in the auctions with and without winning-bids announcement are possible.

First, the first-round expected price in a sequential auction with winning-bids announcement can be lower than the expected first-round price in a sequential auction with no bid announcement. Second, the first-round expected price can be higher than the expected price in a sequential auction with no bid announcement, but lower than the expected price in a single-round auction. In these cases, the revenue comparison between a single-round auction and a sequential auction with winning-bids announcement is ambiguous. Which of the two auction formats yields greater revenue depends on the model parameters. Finally, the expected first-round price in a sequential auction with winning-bids announcement can be higher than the expected price in a single-round auction. In this case, revenue is higher in the sequential auction.

These findings allow us to gain some insights on the comparison between other auction formats and the sequential uniform auction with winning-bids announcement. We know from Milgrom and Weber (1982, 2000) that the ascending (English) auction raises the highest revenue among standard single-round auctions. Note that the single-round ascending auction is equivalent to the single-round uniform auction when there are only three bidders for two objects. Thus, we can conclude that for some model parameters the sequential uniform-price auction with winning-bids announcement raises greater revenue than any standard single-round auction, while for other parameters the ascending auction yields higher revenue.[6]

An additional question, that we do not address in this paper, concerns the optimal seller's choice of how many objects to auction in each round of a sequential auction with winning-bids announcement. A preliminary investigation indicates that there is no general answer: the optimal mix of k_1 and k_2 depends on the model parameters.

5.1 A Numerical Example

We now present the numerical example that shows that revenue and first-round price comparisons are ambiguous. The example is constructed to make the numerical calculations as simple as possible, not to be realistic. We assume that there are three bidders, two objects, each bidder has the same value for one object, and that each bidder's signal is a conditionally independent estimate of this common value. This is a special case of our model in which $n = 3$, $k_1 = k_2 = 1$, $u(V, X_1, X_2, X_3) = V$, and each X_i, $i = 1, 2, 3$, is independently drawn from a conditional density $f(x|v)$. The common value has a discrete distribution: its value

[6] When it raises greater revenue than a single-round ascending auction, the sequential uniform auction with winning-bids announcement also raises greater revenue than a sequential ascending auction because, as shown by Milgrom and Weber (2000), the sequential and the single-round ascending auctions have the same symmetric equilibrium.

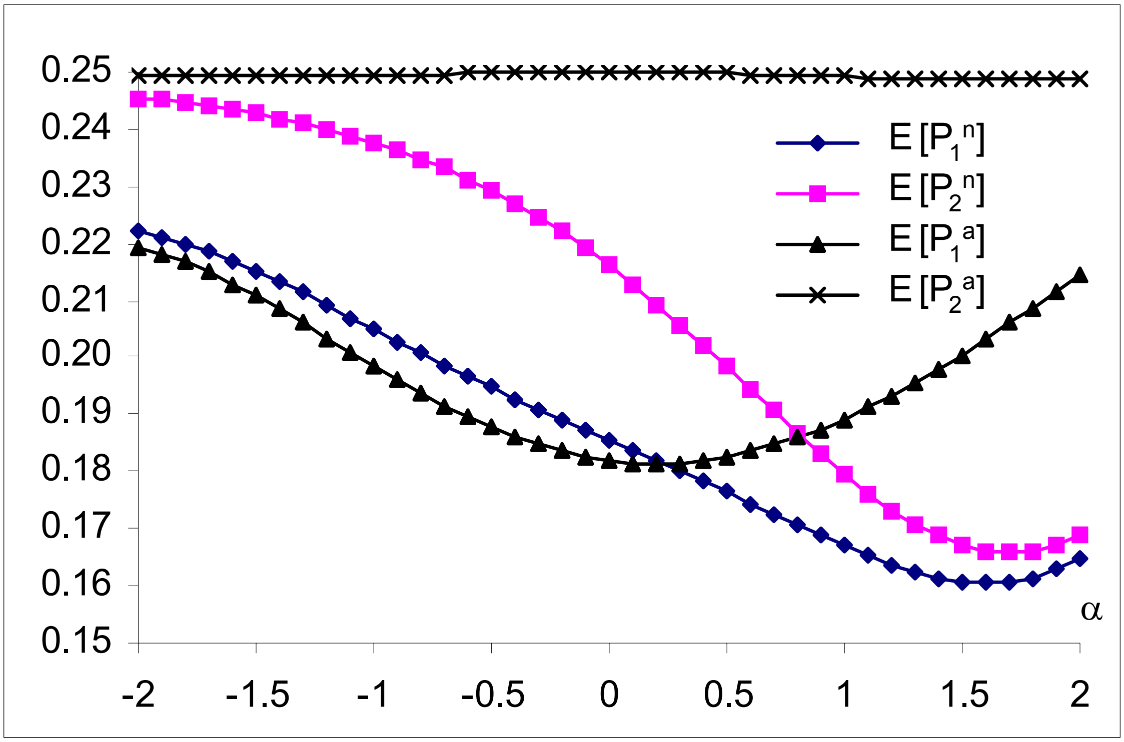

Figure 1: Prices as functions of the parameter α.

is either $v_1 = 0$ or $v_2 = 0.5$, with equal probability. The pdf $f(x|v)$ is given by

$$f(x|v) = \begin{cases} 1 + \alpha (x - v) & \text{if} \quad x \in \left[v - \frac{1}{2}, v + \frac{1}{2}\right], \\ 0 & \text{if} \quad x \notin \left[v - \frac{1}{2}, v + \frac{1}{2}\right], \end{cases} \tag{20}$$

where $\alpha \in [-2, 2]$.[7]

Figure 1 plots the expected prices $E[P_1^n]$, $E[P_2^n]$, $E[P_1^a]$, and $E[P_2^a]$ as functions of α. (Appendix B contains the relevant formulas.) Recall that, by (2) and (3), $E[P_2^n]$ is equal to $E[P^s]$. The following conclusions can be drawn from the figure.

First, by Theorem 3, $E[P_1^n]$ is always less than $E[P_2^n]$, and $E[P_1^a]$ is always less than $E[P_2^a]$. Second, by Theorem 6, $E[P_2^a]$ is always higher than $E[P_2^n]$. Third, $E[P_1^a]$ may

[7]In such a case, the affiliation property can be written as $f(x|v)f(x'|v') \geq f(x|v')f(x'|v)$ for all x, x', v, v' such that $x \geq x'$ and $v \geq v'$. The signal distribution (20) satisfies the affiliation property (1). Note that, although we have assumed that all random variables are continuous, all our results extend to the common-value model in which the common value V has a discrete distribution.

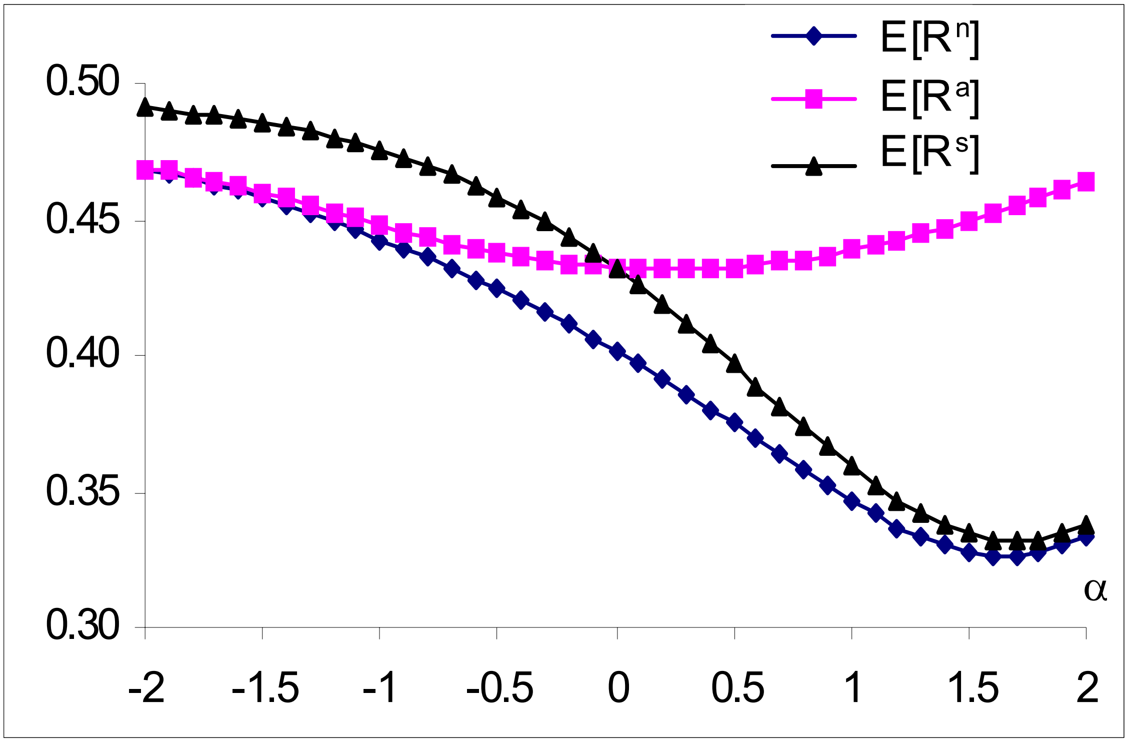

Figure 2: Revenues as functions of the parameter α.

be smaller or greater than $E[P_1^n]$, depending on whether α is smaller or greater than 0.25. Fourth, $E[P_1^a]$ may be smaller or greater than $E[P_2^n]$, depending on whether α is smaller or greater than 0.85.

Figure 2 plots expected auction revenues for the sequential auction with no bid announcement, $E[R^n]$, the sequential auction with winning-bids announcement, $E[R^a]$, and the single-round auction for two objects, $E[R^s]$. As we know from Theorem 4, the expected revenue of a sequential auction with no bid announcement is always lower than the expected revenue of a single-round auction. Figure 2 shows that the expected revenue of a sequential auction with winning-bids announcement may be smaller or greater than the expected revenue of a single-round auction, depending on whether α is smaller or greater than zero.[8]

[8]In our example, for all values of α, expected revenue in a sequential auction is higher with winning-bids announcement than with no bid announcement. In other words, even though I_1^a may be negative, $I_1^a + I_2^a$ turns out to be positive. We have obtained the same result in all other numerical examples we have tried, but we have not been able to provide a formal proof.

5.2 Intermediate Information Policies

We now study the policy of revealing only the lowest first-round winning bid. Such a policy can be viewed as an approximation of the policy of announcing the winning price (the larger the number of bidders, the better the approximation).[9] At the end of this subsection we will also discuss other intermediate information policies.

Let $b_t^\ell(\cdot)$ be the bidding function in round $t = 1, 2$ of a sequential auction with *lowest-winning-bid announcement*. The proof of the following theorem is analogous to the proof of Theorem 2.

Theorem 7. *Let y_{k_1} be the realization of the signal that correspond to the lowest first-round winning bid. A symmetric equilibrium strategy in the sequential uniform auction with lowest-winning-bid announcement is given by*

$$b_2^\ell(x; y_{k_1}) = E\left[V_1 | X_1 = x, Y^{k_1} = y_{k_1}, Y^k = x\right],$$
$$b_1^\ell(x) = E\left[b_2^\ell\left(Y^k; x\right) | X_1 = x, Y^{k_1} = x\right].$$

From the seller's point of view, the policy of only announcing the lowest first-round winning bid is dominated by the policy of revealing all winning bids.

Theorem 8. *In a sequential uniform auction with lowest-winning-bid announcement the expected prices in both rounds are lower than in a sequential uniform auction with winning-bids announcement.*

Proof. Define the random variable P_t^ℓ as the price in round t of a sequential auction with

[9] Announcing the winning price is common in practice. Unfortunately, analyzing such a policy requires dealing with a host of additional technical issues. In the second round, the bid of one of the remaining bidders becomes publicly known. This destroys the symmetry of second-round bidding; see also Milgrom and Weber (2000, pp. 181-182).

lowest-winning-bid announcement. First, note that

$$
\begin{aligned}
&E[b_2^a(z_{k+1};Y^1,...,Y^{k_1-1},z_{k_1})|Y^{k_1}=z_{k_1},X_1=Y^k=z_{k+1}] \\
&\quad = E[E[V_1|X_1=Y^k=z_{k+1},Y^1,...,Y^{k_1-1},Y^{k_1}=z_{k_1}]|Y^{k_1}=z_{k_1},X_1=Y^k=z_{k+1}] \\
&\quad = E[V_1|Y^{k_1}=z_{k_1},X_1=Y^k=z_{k+1}]=b_2^\ell(z_{k+1};z_{k_1}).
\end{aligned}
\tag{21}
$$

The expected second-round price of the sequential auction with winning-bids announcement, conditional on $Z^{k_1}=z_{k_1}$ and $Z^{k+1}=z_{k+1}$, with $z_{k_1}\geq z_{k+1}$, is

$$
\begin{aligned}
E[P_2^a|Z^{k_1}=z_{k_1},Z^{k+1}=z_{k+1}] &= E[P_2^a|Y^{k_1}=z_{k_1}\geq X_1\geq Y^k=z_{k+1}] \\
&= E[b_2^a(z_{k+1};Y^1,...,Y^{k_1-1},z_{k_1})|Y^{k_1}=z_{k_1}\geq X_1\geq Y^k=z_{k+1}] \\
&\geq E[b_2^a(z_{k+1};Y^1,...,Y^{k_1-1},z_{k_1})|Y^{k_1}=z_{k_1}\geq X_1= Y^k=z_{k+1}] \\
&= b_2^\ell(z_{k+1};z_{k_1})=E[P_2^\ell|Z^{k_1}=z_{k_1},Z^{k+1}=z_{k+1}],
\end{aligned}
$$

where the inequality follows from affiliation and the first equality in the last line follows from (21). Taking expectations over Z^{k_1} and Z^{k+1} shows that the expected second-round price is higher when all first-round winning bids are announced than when only the lowest winning bid is announced.

The expected first-round price of the sequential auction with winning-bids announcement, conditional on $Z^{k_1+1}=z_{k_1+1}$, is

$$
\begin{aligned}
E[P_1^a|Z^{k_1+1}=z_{k_1+1}] &= b_1^a(z_{k_1+1})=E[b_2^a(Y^k;Y^1,...,Y^{k_1-1},z_{k_1+1})|X_1=Y^{k_1}=z_{k_1+1}] \\
&= E[E\left[b_2^a(Y^k;Y^1,...,Y^{k_1-1},z_{k_1+1})|X_1=Y^{k_1}=z_{k_1+1}\right]|X_1=Y^{k_1}=z_{k_1+1}] \\
&\geq E[E\left[b_2^a(Y^k;Y^1,...,Y^{k_1-1},z_{k_1+1})|Y^{k_1}=z_{k_1+1},X_1=Y^k\right]|X_1=Y^{k_1}=z_{k_1+1}] \\
&= E[b_2^\ell(Y^k;z_{k_1+1})|X_1=Y^{k_1}=z_{k_1+1}] \\
&= b_1^\ell(z_{k_1+1})=E[P_1^\ell|Z^{k_1+1}=z_{k_1+1}],
\end{aligned}
$$

where the inequality follows from affiliation and the equality in the second to last line follows from (21). Taking expectations over Z^{k_1+1} concludes the proof. ∎

Theorems 7 and 8 can be generalized to other intermediate revelation policies that include revealing the lowest first-round winning bid. The proof of existence of a symmetric equilibrium is analogous to the proof of Theorem 2 and easily extends to multiple rounds.[10] The proof that each of these policies yields a lower expected price in both rounds than the policy of revealing all winning bids is similar to the proof of Theorem 8.

6 Conclusions

We have derived the symmetric equilibrium bidding functions for the sequential uniform auction with and without winning-bids announcement, and we have isolated three effects on revenue of auctioning objects sequentially, rather than simultaneously: a lowballing effect and two informational effects. The lowballing effect reduces bids in the first round. When there are no bid announcements (or values are private), only the lowballing effect is at work and the first-round expected price and the seller's revenue are lower than in a single-round auction. When the first-round winning bids are announced, the second-round informational effect raises the expected second-round price above the price in a single-round auction. On the other hand, the first-round informational effect has an ambiguous impact on first-round expected price. The first-round expected price with winning-bids announcement can range from being lower than with no bid announcement to being higher than the expected price in a single-round auction. As a result, the revenue comparison of a single-round uniform auction and a sequential auction with winning-bids announcement is also ambiguous; either could be higher.

[10] Revealing the lowest winning bid in each round alleviates the difficulty mentioned by Milgrom and Weber (2000, p. 182) of proving that bidding higher, to have better information in subsequent rounds, is not profitable.

Appendix A

Proof of Lemma 1. First, we consider the case where $D(s)$ is a polynomial so that it has a finite number of zeros. After proving the lemma for that case, we generalize the result to any continuous function $D(s)$.

Fix z, $0 < z \leq S$, set $s_0 = 0$, and let $s_1 < s_2 < \cdots < s_{M-1}$ be the zeros of $D(s)$ which are not local extrema and which are smaller than $s_M = z$. Note that $D(s_0) \leq 0$, since otherwise $\int_0^x D(s)a(s)ds > 0$ for small enough x. If M is even, let $2N = M$. By construction, $D(s) \leq 0$ for $s_{2i} \leq s \leq s_{2i+1}$ and $D(s) \geq 0$ for $s_{2i+1} \leq s \leq s_{2i+2}$, with $i = 0, 1, \ldots, N-1$. If M is odd, let $2N = M-1$ and observe that $D(s) \leq 0$ for $s_{M-1} < s \leq s_M$, so that $\int_0^z D(s)ds \leq \int_0^{s_{2N}} D(s)ds$. We will prove that $\int_0^x D(s)a(s)ds \leq 0$ for all x implies that $\int_0^{s_{2N}} D(s)ds \leq 0$.

Since $a(s)$ is positive and non-decreasing, $\int_{s_{2i}}^{s_{2i+1}} D(s)a(s)ds \geq a(s_{2i+1}) \int_{s_{2i}}^{s_{2i+1}} D(s)ds$, and $\int_{s_{2i+1}}^{s_{2i+2}} D(s)a(s)ds \geq a(s_{2i+1}) \int_{s_{2i+1}}^{s_{2i+2}} D(s)ds$. Therefore, for $i = 0, 1, \ldots, N-1$,

$$a(s_{2i+1}) \int_{s_{2i}}^{s_{2i+2}} D(s)ds \leq \int_{s_{2i}}^{s_{2i+2}} D(s)a(s)ds,$$

and so

$$\int_0^{s_{2N}} D(s)ds = \sum_{i=0}^{N-1} \int_{s_{2i}}^{s_{2i+2}} D(s)ds \leq \sum_{i=0}^{N-1} \frac{1}{a(s_{2i+1})} \int_{s_{2i}}^{s_{2i+2}} D(s)a(s)ds.$$

We claim that:

$$\sum_{i=0}^{N-1} \frac{1}{a(s_{2i+1})} \int_{s_{2i}}^{s_{2i+2}} D(s)a(s)ds \leq \frac{1}{a(s_{2N-1})} \int_0^{s_{2N}} D(s)a(s)ds. \tag{22}$$

Note that (22) implies $\int_0^{s_{2N}} D(s)ds \leq \frac{1}{a(s_{2N-1})} \int_0^{s_{2N}} D(s)a(s)ds \leq 0$; that is, if (22) holds then the lemma, under the assumption that $D(s)$ is a polynomial, is proven. The proof of (22) proceeds by induction. Suppose that, for $0 < K < N-1$,

$$\sum_{i=0}^{K-1} \frac{1}{a(s_{2i+1})} \int_{s_{2i}}^{s_{2i+2}} D(s)a(s)ds \leq \frac{1}{a(s_{2K-1})} \int_0^{s_{2K}} D(s)a(s)ds.$$

(Note that the inequality holds for $K = 1$.) Then

$$\begin{aligned}
&\sum_{i=0}^{K-1} \frac{1}{a(s_{2i+1})} \int_{s_{2i}}^{s_{2i+2}} D(s)a(s)ds + \frac{1}{a(s_{2K+1})} \int_{s_{2K}}^{s_{2K+2}} D(s)a(s)ds \\
&\quad \leq \frac{1}{a(s_{2K-1})} \int_{0}^{s_{2K}} D(s)a(s)ds + \frac{1}{a(s_{2K+1})} \int_{s_{2K}}^{s_{2K+2}} D(s)a(s)ds \\
&\quad \leq \frac{1}{a(s_{2K+1})} \int_{0}^{s_{2K}} D(s)a(s)ds + \frac{1}{a(s_{2K+1})} \int_{s_{2K}}^{s_{2K+2}} D(s)a(s)ds.
\end{aligned}$$

The first inequality follows from the induction assumption. The last inequality follows from $\int_0^{s_{2K}} D(s)a(s)ds \leq 0$ and $a(s)$ being positive and increasing, which hold by the lemma's assumptions. This shows that (22) holds and completes the proof of the lemma when $D(s)$ is a polynomial.

Now suppose that $D(s)$ is any continuous function on $[0, S]$. By Weierstrass Approximation Theorem, for any $\varepsilon > 0$ there exists a polynomial P_m of degree m such that $|D(s) - P_m(s)| < \varepsilon$ for all s in $[0, S]$. Define the polynomial $D_m(s) = P_m(s) - \varepsilon$, so that $D_m(s) \leq D(s)$ for all s in $[0, S]$. Thus, if $a(s)$ is a positive function, then $\int_0^x D_m(s)a(s)ds \leq \int_0^x D(s)a(s)ds \leq 0$. Furthermore, there exists a sequence of polynomials D_m such that $lim_{m\to\infty} max_{s\in[0,S]} |D_m(s) - D(s)| = 0$, and hence, for any x, $lim_{m\to\infty} \int_0^x D_m(s)ds = \int_0^x D(s)ds$. We have already shown that the lemma holds for all polynomial functions D_m: for any x, $\int_0^x D_m(s)ds \leq 0$ for all m. As a result, $\int_0^x D(s)ds = lim_{m\to\infty} \int_0^x D_m(s)ds \leq 0$, and the lemma holds for any continuous function $D(s)$. ∎

Appendix B

This appendix presents the detailed derivation of the formulas used for drawing Figure 1 and Figure 2.

It is convenient to work with a slightly more general example than the one in Section 5.1. There are n bidders and two objects. The common value is either v_1 or v_2, with probabilities q_1 and $q_2 = 1 - q_1$, respectively. The signal distribution is

$$f(x|v) = \begin{cases} \dfrac{2 + \alpha(2x - 2v - s_2 + s_1)}{2(s_1 + s_2)} & \text{if} \quad x \in [v - s_1, v + s_2], \\ 0 & \text{if} \quad x \notin [v - s_1, v + s_2], \end{cases}$$

where $\alpha \in \left[-\frac{2}{s_1+s_2}, \frac{2}{s_1+s_2}\right]$, and $v_2 - s_1 \leq v_1 < v_2 \leq v_1 + s_2$ (Figures 1 and 2 assume $n = 3$, $q_1 = q_2 = s_1 = s_2 = 0.5$). Let $\phi(s) = \dfrac{2 + \alpha(2s - s_2 + s_1)}{2(s_1 + s_2)}$, and $\Phi(s) = \int_{-s_1}^{s} \phi(z)dz$.

The joint pdf of bidder 1's signal x, the highest y_1, and second-highest y_2 signals of bidders $2, 3, \ldots, n$, conditional on the common value v, is given by

$$h(x, y_1, y_2|v) = \begin{cases} (n-1)(n-2)f(x|v)f(y_1|v)f(y_2|v)F^{n-3}(y_2|v) & \text{if} \quad y_2 \leq y_1, \\ 0 & \text{if} \quad y_2 > y_1. \end{cases} \tag{23}$$

First, consider the no-bid-announcement case.

From (3), the second-round bidding function is

$$\begin{aligned} b_2^n(x) &= \frac{q_1 \int_x^{\overline{x}} v_1 h(x, y_1, x|v_1)dy_1 + q_2 \int_x^{\overline{x}} v_2 h(x, y_1, x|v_2)dy_1}{q_1 \int_x^{\overline{x}} h(x, y_1, x|v_1)dy_1 + q_2 \int_x^{\overline{x}} h(x, y_1, x|v_2)dy_1} \\ &= \begin{cases} v_1 & \text{if} \quad x < v_2 - s_1 \\ b_{21}^n(x) & \text{if} \quad v_2 - s_1 \leq x \leq v_1 + s_2 \\ v_2 & \text{if} \quad v_1 + s_2 < x, \end{cases} \end{aligned}$$

where

$$b_{21}^n(x) = \frac{q_1 v_1 \phi^2(x - v_1)\Phi^{n-3}(x - v_1)\left(1 - \Phi(x - v_1)\right) + q_2 v_2 \phi^2(x - v_2)\Phi^{n-3}(x - v_2)\left(1 - \Phi(x - v_2)\right)}{q_1 \phi^2(x - v_1)\Phi^{n-3}(x - v_1)\left(1 - \Phi(x - v_1)\right) + q_2 \phi^2(x - v_2)\Phi^{n-3}(x - v_2)\left(1 - \Phi(x - v_2)\right)}.$$

Let

$$f_3(x|v) = \frac{n(n-1)(n-2)}{2}\phi(x-v)\Phi^{n-3}(x-v)(1-\Phi(x-v))^2.$$

Then the expected second-round price is

$$\begin{aligned} E\left[P_2^n\right] = E\left[b_2^n(Z^3)\right] &= q_1 v_1 \int_{v_1-s_1}^{v_2-s_1} f_3(x|v_1)dx + q_1 \int_{v_2-s_1}^{v_1+s_2} b_{21}^n(x) f_3(x|v_1)dx \\ &\quad + q_2 \int_{v_2-s_1}^{v_1+s_2} b_{21}^n(x) f_3(x|v_2)dx + q_2 v_2 \int_{v_1+s_2}^{v_2+s_2} f_3(x|v_2)dx. \end{aligned}$$

Let

$$\phi_2(x, y_2|v) = (n-2)\phi^2(x-v)\phi(y_2-v)\Phi^{n-3}(y_2-v). \tag{24}$$

By (4) using (23), the first-round bidding function is

$$\begin{aligned} b_1^n(x) &= \frac{q_1 \int_{\underline{x}}^{x} b_2^n(y_2) h(x,x,y_2|v_1)dy_2 + q_2 \int_{\underline{x}}^{x} b_2^n(y_2) h(x,x,y_2|v_2)dy_2}{q_1 \int_{\underline{x}}^{x} h(x,x,y_2|v_1)dy_2 + q_2 \int_{\underline{x}}^{x} h(x,x,y_2|v_2)dy_2} \\ &= \begin{cases} v_1 & \text{if} \quad x < v_2 - s_1 \\ b_{11}^n(x) & \text{if} \quad v_2 - s_1 \le x \le v_1 + s_2 \\ b_{12}^n(x) & \text{if} \quad v_1 + s_2 < x, \end{cases} \end{aligned}$$

where

$$b_{11}^n(x) = \frac{q_1 v_1 \int_{v_1-s_1}^{v_2-s_1} \phi_2(x,y_2|v_1)dy_2 + q_1 \int_{v_2-s_1}^{x} b_{21}^n(y_2)\phi_2(x,y_2|v_1)dy_2 + q_2 \int_{v_2-s_1}^{x} b_{21}^n(y_2)\phi_2(x,y_2|v_2)dy_2}{q_1\phi^2(x-v_1)\Phi^{n-2}(x-v_1) + q_2\phi^2(x-v_2)\Phi^{n-2}(x-v_2)},$$

$$b_{12}^n(x) = (n-2)\frac{\int_{v_2-s_1}^{v_1+s_2} b_{21}^n(y_2)\phi(y_2-v_2)\Phi^{n-3}(y_2-v_2)dy_2 + v_2\int_{v_1+s_2}^{x}\phi(y_2-v_2)\Phi^{n-3}(y_2-v_2)dy_2}{\Phi^{n-2}(x-v_2)}.$$

Let

$$\phi_1(x|v) = n(n-1)\phi(x-v)\Phi^{n-2}(x-v)(1-\Phi(x-v)). \tag{25}$$

Then the expected first-round price is

$$E\left[P_1^n\right] = E[b_1^n(Z^2)] = q_1 v_1 \int_{v_1-s_1}^{v_2-s_1} \phi_1(x|v_1)dx + q_1 \int_{v_2-s_1}^{v_1+s_2} b_{11}^n(x)\phi_1(x|v_1)dx$$
$$+ q_2 \int_{v_2-s_1}^{v_1+s_2} b_{11}^n(x)\phi_1(x|v_2)dx + q_2 \int_{v_1+s_2}^{v_2+s_2} b_{12}^n(x)\phi_1(x|v_2)dx.$$

Now, consider the winning-bids-announcement case.

The second-round bidding function is

$$\begin{aligned} b_2^a(x;y_1) &= \frac{q_1 v_1 h(x,y_1,x|v_1) + q_2 v_2 h(x,y_1,x|v_2)}{q_1 h(x,y_1,x|v_1) + q_2 h(x,y_1,x|v_2)} \\ &= \begin{cases} v_1 & \text{if} \;\; x < v_2 - s_1 \\ b_{21}^a(x,y_1) & \text{if} \;\; v_2 - s_1 \le x \le y_1 \le v_1 + s_2 \\ v_2 & \text{if} \;\; v_1 + s_2 < y_1, \end{cases} \end{aligned}$$

where

$$b_{21}^a(x,y_1) = \frac{q_1 v_1 \phi^2(x-v_1)\Phi^{n-3}(x-v_1)\phi(y_1-v_1) + q_2 v_2 \phi^2(x-v_2)\Phi^{n-3}(x-v_2)\phi(y_1-v_2)}{q_1 \phi^2(x-v_1)\Phi^{n-3}(x-v_1)\phi(y_1-v_1) + q_2 \phi^2(x-v_2)\Phi^{n-3}(x-v_2)\phi(y_1-v_2)}.$$

Let

$$\phi_3(x,y_1|v) = n(n-1)(n-2)\phi(x-v)\phi(y_1-v)\Phi^{n-3}(x-v)\left[\Phi(y_1-v) - \Phi(x-v)\right].$$

Then the expected second-round price is

$$\begin{aligned} E\left[P_2^a\right] &= E[b_2^a(Z^3, Z^1)] \\ &= q_1 v_1 \int_{v_1-s_1}^{v_2-s_1} \int_x^{v_1+s_2} \phi_3(x,y_1|v_1)dy_1 dx + q_2 v_2 \int_{v_1+s_2}^{v_2+s_2} \int_{v_2-s_1}^{y_1} \phi_3(x,y_1|v_2)dx dy_1 \\ &+ q_1 \int_{v_2-s_1}^{v_1+s_2} \int_x^{v_1+s_2} b_{21}^a(x,y_1)\phi_3(x,y_1|v_1)dy_1 dx + q_2 \int_{v_2-s_1}^{v_1+s_2} \int_{v_2-s_1}^{y_1} b_{21}^a(x,y_1)\phi_3(x,y_1|v_2)dx dy_1. \end{aligned}$$

By (6), the first-round bidding function is

$$
\begin{aligned}
b_1^a(x) &= \frac{q_1 \int_{\underline{x}}^{x} b_2^a(y_2,x) h(x,x,y_2|v_1) dy_2 + q_2 \int_{\underline{x}}^{x} b_2^a(y_2,x) h(x,x,y_2|v_2) dy_2}{q_1 \int_{\underline{x}}^{x} h(x,x,y_2|v_1) dy_2 + q_2 \int_{\underline{x}}^{x} h(x,x,y_2|v_2) dy_2} \\
&= \begin{cases} v_1 & \text{if} \quad x < v_2 - s_1 \\ b_{11}^a(x) & \text{if} \quad v_2 - s_1 \leq x \leq v_1 + s_2 \\ v_2 & \text{if} \quad v_1 + s_2 < x, \end{cases}
\end{aligned}
$$

where, using definition (24),

$$
b_{11}^a(x) = \frac{q_1 v_1 \int_{v_1-s_1}^{v_2-s_1} \phi_2(x,y_2|v_1) dy_2 + q_1 \int_{v_2-s_1}^{x} b_{21}^a(y_2,x) \phi_2(x,y_2|v_1) dy_2 + q_2 \int_{v_2-s_1}^{x} b_{21}^a(y_2,x) \phi_2(x,y_2|v_2) dy_2}{q_1 \phi^2(x-v_1) \Phi^{n-2}(x-v_1) + q_2 \phi^2(x-v_2) \Phi^{n-2}(x-v_2)}.
$$

Using definition (25), the expected first-round price is

$$
\begin{aligned}
E\left[P_1^a\right] = E[b_1^a(Z^2)] = q_1 v_1 \int_{v_1-s_1}^{v_2-s_1} \phi_1(x|v_1) dx + q_1 \int_{v_2-s_1}^{v_1+s_2} b_{11}^a(x) \phi_1(x|v_1) dx \\
+ q_2 \int_{v_2+s_1}^{v_1+s_2} b_{11}^a(x) \phi_1(x|v_2) dx + q_2 v_2 \int_{v_1+s_2}^{v_2+s_2} \phi_1(x|v_2) dx.
\end{aligned}
$$

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