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Multidimensional Mechanism Design: Revenue Maximization and the Multiple-Good Monopoly

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Abstract

The seller of N distinct objects is uncertain about the buyer's valuation for those objects. The seller's problem, to maximize expected revenue, consists of maximizing a linear functional over a convex set of mechanisms. A solution to the seller's problem can always be found in an extreme point of the feasible set. We identify the relevant extreme points and faces of the feasible set. With $N = 1$, the extreme points are easily described providing simple proofs of well-known results. The revenue-maximizing mechanism assigns the object with probability one or zero depending on the buyer's report. With $N > 1$, extreme points often involve randomization in the assignment of goods. Virtually any extreme point of the feasible set maximizes revenue for a well-behaved distribution of buyer's valuations. We provide a simple algebraic procedure to determine whether a mechanism is an extreme point.

Keywords: extreme point, exposed point, faces, non-linear pricing, monopoly pricing, multi-dimensional, screening, incentive compatibility, adverse selection, mechanism design.

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1 Introduction

The seller of N distinct, indivisible objects wishes to maximize the revenue from selling the objects. There is a single potential buyer. The seller has prior beliefs about the buyer's valuations, the maximum amounts that the buyer is willing to pay for each object. The buyer's valuation for any set of objects is the sum of his valuations for the individual goods in the set.

The seller's problem is an optimization program where the variables are functions—i.e, the feasible mechanisms—and where the objective function—i.e, the seller's expected revenue—is linear on the mechanism. The set of all feasible mechanisms is convex and compact, and hence it has extreme points. As in finite-dimensional programs, the set of maximizers of the linear objective function coincides with a face of the feasible set. Furthermore, a maximizer can always be found at an extreme point of the feasible set (Bauer Maximum Principle). By characterizing the relevant faces and extreme points of the feasible set, we identify the potential solutions to the seller's problem.

Much of the theory and most applications of mechanism design are concerned with one-dimensional environments. For instance, environments where the seller has a single object to sell and the uncertainty variable is the buyer's valuation. The solution to one-dimensional problems is well understood. When the seller has more than one object to sell and the uncertainty variable represents the buyer's valuations for each good, the problem becomes multi dimensional. Even when the buyer's valuations for each good are independently distributed, it is not known, in general, how the seller should conduct the sale in order to maximize revenue. Although for some prior distributions of valuations, the optimal mechanism has been identified, general results remain elusive.

Perhaps a reason for this elusiveness is that the structure of the feasible set in the seller's problem changes significantly when there are more than one good to sell. In one-dimensional environments, all extreme points of the feasible set have the same form. When a mechanism is represented by a function $p(x)$ indicating the probability that a buyer with reported valuation x will get the object, extreme points are step functions with at most two steps. In one step, the good is not traded ($p(x) = 0$); in the other one the good is traded for certain ($p(x) = 1$). (We call these mechanisms zero-one.) The seller can implement the extreme point by posting an appropriate price for the good, and consumer types screen themselves into those types who purchase the object and those who do not. This implies that to maximize expected revenue, it is never necessary to randomize in the assignment of the object. The prior distribution of buyer's valuations determines the actual price posted but not the general form of the mechanism. Zero-one mechanisms maximize expected seller's revenue for any prior distributions of buyer's valuations.

The higher-dimensional analogue of a zero-one mechanism potentially includes a step where no goods are traded, various steps where a subset of objects is traded for certain but the remaining objects are not traded at all, and a step where all goods are traded. Once again the seller can implement the zero-one mechanism by posting prices, a price for each individual good and a price for each possible bundle. We find that in higher-dimensional environments, the set of extreme points contains many "novel" mechanisms, mechanisms that are not zero-one. In particular, extreme points need not be step functions (Example 2), and even when they are, they may have steps where objects are randomly assigned to consumers (Examples 1 and 3). *Posting prices, even bundle prices, no longer suffices to implement the extreme point mechanisms.* The prior distribution of buyer's valuations, an unobservable element of the model, now determines the *form* of the optimal mechanism, a notable difference with the one-dimensional environment.

In optimization programs of the sort we study, there is always an extreme point that minimizes the objective function. For instance, the mechanism that never sells the good is an extreme point, it generates no revenue, and it is a minimum for all prior valuations. One might conjecture that the "novel" extreme points illustrated by our examples are within the class of mechanisms that never maximize expected revenue. We show that this is not the case. Any mechanism specifies the dollar amount $t(x)$ that a buyer of valuation x must transfer to the seller; i.e. $t(x)$ is the seller's revenue of dealing with a buyer of valuation x under the mechanism. We say a mechanism is undominated if there is no alternative mechanism that generates at least as much seller's revenue for all buyer's valuations (and strictly more for some). We prove that *every* undominated mechanism—not just the extreme points—maximizes expected seller's revenue for some independent distribution of valuations (Theorem 3). This describes the relevant portion of the boundary of the feasible set. We also show that all our "novel" examples of extreme points are, indeed, undominated (Corollary 1 and Remark 7).

Within the extreme points of the feasible set, those that are step functions have a salient feature. They divide the different buyer's types into finitely many groups; buyers within each group are treated equally by the mechanism in the sense that they get the goods with the same probability, and pay the same amount to the buyer. We demonstrate that the set of extreme points that are step functions is norm dense in the set of *all* extreme points (Corollary 2). Since expected seller's revenue is always maximized at an extreme point, there is little loss in restricting attention to *step* mechanisms.

One might have hoped that "novel" extreme points might be peculiar, in the sense that they are not plentiful. This, we show, is not the case. We find an algebraic procedure, based on a characterization of some relevant faces of the feasible set, to determine whether a proposed step mechanism is an extreme point (Theorem 8 and Subsection 6.3). In our procedure, determining whether a step mechanism is an extreme point is, essentially, equivalent to determining if a consistent, linear system of finitely many equations has a unique solution. If the coefficient matrix has full rank the mechanism is an extreme point. It follows easily using this procedure that step mechanisms are generically extreme points. For instance, the "novel" extreme points in our examples are generically so, in the sense that small changes in the steps will not alter their status as extreme points.

We note that our methods have a strong geometric quality and are elementary, modulus some definitions from functional analysis.

In realistic applications, the variable representing private information is likely to have more than one dimension. Furthermore, the optimal mechanism in multi-dimensional environments may be qualitatively very different than the optimal mechanism in similar one-dimensional environments. Thus, contributions to the theory of multi-dimensional mechanism appear important.

We conclude the introduction with a brief review of related literature. Our primary concern is with the theory of mechanism design in multi-dimensions. The application on which we focus, that of monopoly pricing, is of independent interest and has a long history in economics. Adams and Yellen (1976) showed by example that if the buyer's valuations are negatively correlated, the monopolist may obtain higher revenue by bundling—posting a bundle price in addition to prices for the individual goods. McAfee, McMillan, and Whinston (1989) provide sufficient conditions under which bundling dominates individual sales, and note that when the buyer's valuations are independently distributed those conditions are satisfied. More recently, Fang and Norman (2003) compared the relative virtues of two particular mechanisms when prior beliefs are log-concave.

None of the papers mentioned poses a full mechanism design question in the sense that they restrict a priori the seller's available instruments. The multi-dimensional mechanism design literature is not as extensive. We will present a summary here. (Rochet and Stole (2003) offer a very readable and comprehensive survey.) Different authors use slightly different models and assumptions; the interested reader should consult the original sources.

Rochet (1985) shows among other things that, as in one-dimensional environments, a mechanism is incentive compatible if and only if the buyer's utility induced by the mechanism is convex. This characterization has been extensively used. We discuss it in more detail when we set up the model. McAfee and McMillan (1988) propose a generalized "single crossing property" to pursue global optima. They use this condition to extend the results of Laffont, Maskin, and Rochet (1985) in a model with a single good but where consumers are differentiated by a two-dimensional parameter.

Wilson (1993) derives first order-conditions for the optimality of a mechanism. In general this approach does not yield a description of the optimal mechanism. Wilson also uses computational methods to obtain particular solutions. Armstrong (1996) extends the one-dimensional methodology. He obtains a general and useful principle, his "exclusion" principle. He proves that when there are at least two objects to sell, provided the support of the prior density of buyer's valuations is strictly convex, the optimal mechanism will assign no goods to a group of buyers of positive measure. This is important because the same result does not hold in the one-dimensional case. In addition, Armstrong finds closed-form solutions in some environments where the only binding incentive compatibility constraints are along rays from the origin. As Rochet and Choné (1998) explain, the assumptions necessary to obtain these environments make them the exception rather than the rule. Armstrong (1999) studies how to find an approximately optimal mechanism in certain models when the number of objects to be sold is large.

Rochet (1995) and Rochet and Choné (1998) analyze a general multi-dimensional screening model. Their work represents the state of the art on the subject. In their setting, binding incentive compatibility constraints are not known a priori. They show that, in general, the monopolist will use mechanisms in which there is bunching, i.e., different consumer-types will be treated equally.

Rochet and Choné develop a methodology for dealing with bunching in multi-dimensions. Basov (2001) extends Rochet and Choné's "sweeping" technique using a Hamiltonian approach.

Thanassoulis (2004) studies a model with two perfectly substitutable goods and shows, among other things, that randomization in the assignment of goods typically dominates deterministic assignments.

One may hope that restricting the class of prior distributions may yield some general results.[1] Our work suggests that if the class of prior densities considered is sufficiently rich, so will be the variety of solutions to the seller's problem. As a point of methodology, we think it may be more promising to proceed in the opposite direction, that is to say, to propose a class of mechanisms and then to find the prior densities under which those mechanisms solve the seller's problem. This is what we do in a companion paper (Manelli and Vincent (2004a)). There we search for the prior distributions for which the zero-one mechanisms, i.e. the posting of prices for bundles, are the solution to the seller's problem. We identify general sufficient conditions. Given the duality methods used, we believe the conditions are tight.

There have also been some recent contributions to the question of optimal multiple-object auctions. We mentioned only two, Kazumori (2001) and Zheng (2000). (The interested reader should consult the references listed by them.) Kazumori applies the Rochet and Choné's sweeping procedure. Zheng adapts many of the ideas in Armstrong (1996). He also obtains an explicit formula for the non-linear pricing mechanism in his setting.

Section 2 presents the basic notation and describes the model. Section 3 describes the optimization program in terms of the buyer's indirect utility. Section 4 contains examples, both single and multi-dimensional. Section 5 describes the class of mechanisms that maximize the seller's expected revenue. Section 6 studies the faces and extreme points of the feasible set. It also introduces a procedure to determine whether a mechanism is an extreme point. All Lemmas referenced through out the paper are located in the Appendix.

2 Preliminaries

2.1 Notation

Given any real vector space X, and elements $x, x' \in X$, we let $[x, x'] = \{\alpha x + (1-\alpha)x' : \alpha \in [0,1]\}$. A sequence in X is denoted by $\{x_n\} \in X$; when confusion is unlikely we may use x_n to denote both the sequence and its n^{th} element.

Given a subset E of a topological space X, $int\, E$ is the interior of E and $\overline{E}$ is the closure of E.

We let I represent the interval $[0,1]$. For any positive integer N, a *ray* from the origin through an element $x \in I^N$ is defined as $\mathcal{R}_x = \{\delta x : \delta \in [0,\infty)\}$. We denote by $\mathbb{R}^N_+$ and $\mathbb{R}^N_-$ the weakly

[1] Thanassoulis (2004) shows that conditions on prior beliefs, previously believed to guarantee that zero-one mechanisms maximize seller's expected revenue, do not do so. (Manelli and Vincent (2004a) independently provided another example in this regard.)

positive and weakly negative orthants of $\mathbb{R}^N$. To avoid confusion, we write $\mathbf{0}$ to denote the null element in $\mathbb{R}^N$, and $\mathbf{1}$ to denote $(1, 1, \ldots, 1)$ in $\mathbb{R}^N$. The i^{th} component of any vector $x \in \mathbb{R}^N$ is denoted by x_i; x_{-i} is the vector obtained by removing x_i from x; and (y, x_{-i}) is the vector constructed by replacing x_i in the vector x with $y \in \mathbb{R}$.

Given $A \subset \mathbb{R}^N$, $\mathtt{1}_A$ is the indicator function of A.

The Lebesgue measure is denoted by λ. For $1 \leq p \leq \infty$, $L^p(I^N)$ is the classical Banach space of equivalent classes of real-valued functions f on I^N with finite norm $\|f\|_p$. We will often write simply L^p. If $f \in L^p$ and g is an element of its dual L^q, then the bilinear dual operation is denoted by $\langle f, g \rangle = \int_{I^N} f(x)g(x)\, d\lambda$.

Let u be a real-valued function defined on a subset E of $\mathbb{R}^N$. Then, for all x in E, $u_+(x) = \max\{u(x), 0\}$, and $u_-(x) = \max\{-u(x), 0\}$. If u is differentiable at x, its gradient at x is denoted by $\nabla u(x)$ in $\mathbb{R}^N$; $\nabla_i u(x)$ is its i^{th} component.

We adopt the convention that division by zero is infinity.

2.2 Model

Throughout this essay we assume the following environment. A seller with N different objects faces a single buyer whose valuations are private information. The buyer's preferences over consumption and money transfers are given by $U(x, q, t) = x \cdot q - t$, where $x \in I^N$ is the N-vector of buyer's valuations, q is the vector of quantities consumed of each good, and $t \in \mathbb{R}$ is a monetary transfer made to the seller. The buyer's valuation x is distributed according to a density function $f(x)$ that represents the seller's beliefs about the buyer's private information.

The seller's problem is to design a revenue-maximizing mechanism to carry out the sale. By the revelation principle, the seller may restrict attention to direct revelation mechanisms where each buyer type reports his type truthfully.[2] A direct revelation mechanism is a pair of integrable functions

$$p : I^N \longrightarrow I^N$$
$$t : I^N \longrightarrow \mathbb{R},$$

where, given the buyer's valuation x, $p_i(x)$ (i.e. the i^{th} component of $p(x)$) is the probability that the buyer will obtain good i given her valuation x, and $t(x)$ is the transfer made by the buyer to the seller.[3]

The buyer's expected payoff $u(x'|x)$ under the direct revelation mechanism (p, t), when the buyer has valuation x and reports x' is $u(x'|x) = p(x') \cdot x - t(x')$. We define

$$u(x) = p(x) \cdot x - t(x).$$

[2] The possibility of multiple equilibria in the direct revelation mechanism is the basis of a well-known critique to the use of the revelation principle.

[3] In an alternative formulation $p(x)$ could be an arbitrary probability measure on I^N, thus allowing for correlation. Since the buyer's preferences are linear there is no loss of generality in the alternative we adopted in the paper.

The buyer must prefer to reveal its information truthfully—incentive compatibility (IC)—and to participate in the mechanism voluntarily—individual rationality (IR). Thus (p, t) satisfies IC and IR if and only if

(IC)	for almost all $x, u(x) \geq u(x'\|x) \; \forall x'$
(**IR**)	for almost all $x, u(x) \geq 0$.

The seller's problem is therefore to select the functions (p, t) to maximize expected revenue, $E(t)$, subject to IC and IR.

3 The Program

When $N = 1$, the seller's problem is usually simplified using Myerson's characterization of incentive compatibility: a mechanism satisfies IC if and only if p is non-decreasing. In turn, integrating a non-decreasing p, one obtains the buyer's expected payoff $u(x) = u(0) + \int_0^x p(y)\, dy$. The seller's problem is then generally stated and solved using only the probability-of-trade function p.

We set up the optimization problem in two different formats. In the first one, we use the payoff functions u as the variable of optimization. In the second one we use the transfer functions t as the optimization variables. A useful characterization of incentive compatibility first noted by Rochet (1985) gives us the choice.[4] We first present this characterization as Theorem 1. Then we set the program in terms of the buyer's payoffs u. In Section 5 we formulate the program in terms of the transfer functions t. We use both formulations throughout the paper.

Theorem 1 (Rochet) *If (p, t) satisfies IC, the corresponding buyer's expected payoff $u(x)$ is convex with gradient $\nabla u(x)$ in I^N for most x. Indeed $\nabla u(x) = p(x)$ almost everywhere.*

If $u(x)$ is a convex function with gradient $\nabla u(x)$ in I^N for most $x \in I^N$, then there exist functions (p, t) satisfying IC such that u represents the corresponding buyer's payoffs. The direct revelation mechanism is defined by $p(x) = \nabla u(x)$ almost everywhere, and $t(x) = \nabla u(x) \cdot x - u(x)$.

The theorem states that, roughly, a mechanism is IC if and only if the corresponding buyer's payoff is convex, with partial derivatives between zero and one. The proof, which we omit, is based on the following observations. Given a direct revelation mechanism (p, t), $u(x)$ is, because of IC, the supremum of a family of linear functions, i.e., $\sup\{[p(x') \cdot x - t(x')] : x' \in I^N\}$. The supremum of such a family is convex, hence u is convex. Note that $\nabla u(x) = p(x)$. We turn to the converse. Given a convex function u with gradient in I^N, the direct revelation mechanism (p, t) is easily recovered: the probability of trade $p(x)$ is the "slope" of the hyperplane tangent to the graph of u at the point $(x, u(x))$, and $-t(x)$ is the intercept of such hyperplane.

Theorem 1 characterizes incentive compatibility. We wish the mechanism to satisfy also individual rationality. Thus u must be non-negative. Since the objective is to find an optimal policy

[4] This characterization has been extensively used in the literature. See, for instance, Armstrong (1996), and Jehiel, Moldovanu, and Stacchetti (1998).

for the seller, and since the buyer's expected payoff is non-decreasing, any mechanism that maximizes expected seller's revenue will yield payoff $u(\mathbf{0}) = 0$ to buyers with valuation $x = \mathbf{0}$. Thus when maximizing expected seller's revenue, we may restrict attention to those mechanisms. This discussion justifies the following definition.

Definition 1 *The feasible set in the seller's problem is*

$$W = \left\{ u \in C^0(I^N) \mid u(x) \textit{ is convex, } \nabla u(x) \in I^N \textit{ a.e., and } u(\mathbf{0}) = 0 \right\}.$$

A feasible mechanism is any element u of W.

Given a feasible mechanism u, a buyer with type x receives $u(x) = \nabla u(x) \cdot x - t(x)$. The seller's revenue from a buyer of type x when using mechanism u is $t(x) = \nabla u(x) \cdot x - u(x)$. The seller's expected revenue is therefore $E[t(x)] = E\left[\nabla u(x) \cdot x - u(x)\right]$. Hence, the seller's problem is

$$\max_{u \in W} E[\nabla u(x) \cdot x - u(x)]. \tag{1}$$

The objective function of the seller's problem is an expectation and is linear on the optimization variable, the function u in problem (1).

Note that any u obtained as a convex combination of elements of W is convex, non-negative, and satisfies the bounds on partial derivatives (its gradient takes values in I^N). Hence, W is itself a convex set. It is also simple to verify that W is compact with respect to the sup-norm topology (Lemma 2). Thus, the seller wishes to maximize a linear function on a convex compact set.

If such maximization took place on the plane, the solution would be at a point where a hyperplane representing a level set of the objective function is tangent to the feasible set. The solution set may be a singleton or it may include a segment. If the feasible set were a polygon, the solution set would always include a corner although it might also include an entire face of the polygon. The intuition derived from the plane carries over to our infinite dimensional optimization problem.

We now provide the appropriate definitions of "face" and "corner" in our setting.

Definition 2 *Let V be a subset of a linear space X. A set $E \subset V$ is an extreme set of V if*

$$[(x = \alpha y + (1-\alpha) z) \in E,\ \alpha \in (0,1),\ z, y \in V] \implies y, z \in E.$$

A face is an extreme set of V that is also convex. An extreme set of V consisting of a single point is an extreme point of V.

Thus a point $u \in V$ is an extreme point of V if for every $g \in X$ with $g \neq 0$, $u + g$ does not belong to V or $u - g$ does not belong to V. Alternatively, $u \in V$ is an extreme point of V if u is not the midpoint of any segment included in V.

The Bauer Maximum Principle (Lemma 4) implies that *the set of maximizers in the seller's problem must be a face of the feasible set; and that the maximum is achieved at an extreme point* *of the feasible set.* A characterization of the extreme points of W will prove useful in analyzing the seller's problem.

In the following Section, as an illustration, we apply this methodology to the well-known case of the single-good monopolist. We also show there by example, that the conclusions derived when $N = 1$ do not hold for multiple-good monopolists.

4 Examples

We illustrate the difference between the single-good and the multiple-good monopoly. We first characterize the extreme points of W when $N = 1$. We then show, by example, that when $N > 1$, the extreme points of W have very different characteristics.

4.1 One-Dimensional Case

The following theorem characterizes the extreme points of W.

Theorem 2 *If the seller has a single good, a mechanism $u \in W$ is an extreme point if and only if for most buyer's valuations $x \in I^N$, the object is assigned either with probability one or with probability zero, i.e., $p(x) = \nabla u(x) \in \{0, 1\}$.*

Proof Let $u \in W$ be such that $\nabla u(x) \in \{0, 1\}$ for for almost all $x \in I$. Let g be any real-valued function defined on I^N. If g is not continuous, or a.e. differentiable, then $u + g$ is not in W. If $\nabla g(x) \neq 0$ a.e., then $u + g$ or $u - g$ are not in W. Hence, $\nabla g(x) = 0$ a.e, and therefore $g = 0$. We conclude u is an extreme point of W.

To establish the converse select any $u \in W$ that is not a zero-one mechanism. Then, there is a set of positive measure $B \subset [0, 1]$ such that $\epsilon < \nabla u(x) < 1 - \epsilon$. Let

$$\nabla g(x) = \begin{cases} 1 - \nabla u(x) & \text{if } \nabla u(x) > 0.5 \\ \nabla u(x) & \text{if } \nabla u(x) \leq 0.5 \end{cases}$$

Let $g(x) = \int_0^x \nabla g(z)dz$; then $g(x)$ is a continuous function. We now verify that both $u + g$ and $u - g$ are in W. First, the gradient of $u + g$ is in $[0, 1]$:

$$\nabla(u(x) + g(x)) = \begin{cases} 1 & \text{if } \nabla u(x) > 0.5 \\ 2\nabla u(x) & \text{if } \nabla u(x) \leq 0.5 \end{cases}$$

Second, since $\nabla(u(x) + g(x))$ is increasing in x, $u + g$ is convex. Third, $g(0) = 0$ by construction. Thus $u + g$ is in W. A similar argument applies to $u - g$.

Q.E.D.

Theorem 2's characterization of the extreme points of W immediately provides an alternative and simple proof of various well-known results which we summarize below (Myerson 1981).

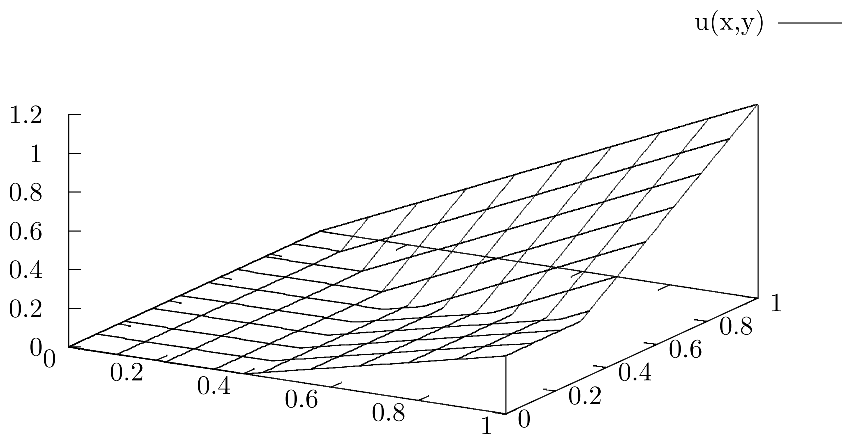

Figure 1: $u(x) = \max\{0,\ (0.5x_1 - 0.2),\ (x_1 + x_2 - 1)\}$

1. A take-it-or-leave-it offer is the mechanism that maximizes expected seller's revenue among all feasible bargaining mechanisms. (Given the optimal mechanism, p, the offer is $\inf\{x \in [0,1] : p(x) = 1\}$.)

2. Randomization (i.e. "haggling" as in Riley and Zeckhauser (1984)) is not necessary to maximize expected seller's revenue.

3. The revenue-maximizing mechanism is piecewise linear, i.e. the buyer's expected payoff u is a piecewise linear function. The transfer t and the probability of trade p are step functions with at most two steps.

The next Subsection illustrates with several examples that these well-known results do not extend to higher dimensions.

4.2 Some Two-Dimensional Examples

Two examples demonstrate that the set of extreme points of W in higher dimensions is considerably richer than in one dimension. In both examples there are two goods ($N = 2$). The first example identifies an extreme point of W that is piecewise linear but involves randomization. The second example identifies an extreme point that is not piecewise linear.

Not all extreme points are necessarily a solution to a well posed seller's problem. In Section 5 we show that the extreme points found in our Examples are the optimal mechanisms for some well-behaved prior distribution of bidders' types.

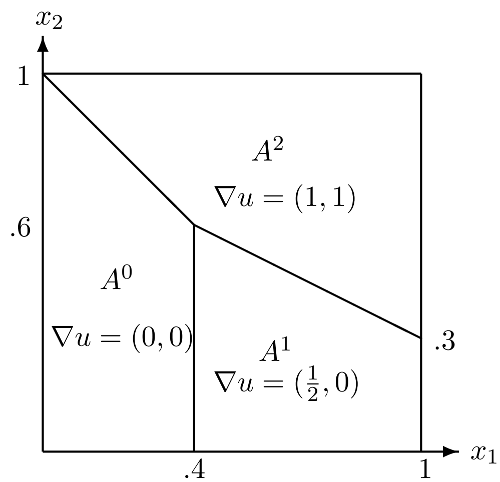

Figure 2: Market Segments

Example 1 *An extreme point with random assignments. Let $N = 2$ and let $u \in W$ be*

$$u(x) = \max\{0,\ (0.5x_1 - 0.2),\ (x_1 + x_2 - 1)\}.$$

The graph of u is depicted in Figure 1. The mechanism u has three linear pieces; it defines three pieces, $A^0 = \{x \in I^N : \nabla u(x) = (0,0)\}$, $A^1 = \{x \in I^N : \nabla u(x) = (\frac{1}{2},0)\}$, and $A^2 = \{x \in I^N : \nabla u(x) = (1,1)\}$, depicted in Figure 2. Buyers with valuations in any given piece are treated similarly. Consider, for instance, a buyer with valuation $x \in A^1$. The corresponding probabilities of trade are $\nabla u = (\frac{1}{2},0)$. The buyer never receives good two. The toss of a fair coin determines whether the buyer receives good one. [5]

We now show that u is indeed an extreme point of W. If u is not an extreme point, then there is a function $g \neq 0$ such that both $u+g$ and $u-g$ are in W. Since both functions $u \pm g$ are continuous, and a.e. differentiable (Lemma 1(iii) and (iv)), g must be continuous, and a.e. differentiable. Then for $x \in A^0 \cup A^2$, ∇g must be identically zero; otherwise either $\nabla(u+g)$ or $\nabla(u-g)$ is not in I^N. It follows that $g(x) = 0$ for all $x \in A^0 \cup A^2$. If $g(x) > 0$ for some $x \in A^1$, then since $u + g$ is non-decreasing (Lemma 1(ii)) , $g(x') > 0$ for some $x' \in A^2 \cap A^1$. This is a contradiction. (A similar argument applies to $g(x) < 0$ using $u - g$.)

In one-dimensional environments, randomizing mechanisms are never extreme points. They may still however be optimal. Geometrically whenever a randomizing mechanism is optimal, it belongs to a face of the feasible set W. Thus the same expected revenue can always be achieved with a non-randomizing mechanism corresponding to a vertex of the face. Example 1 shows that in higher dimensions, there are extreme points that involve randomization.

[5] Note that the direct mechanism can be implemented with an indirect mechanism that consists of a menu of choices offering a fifty percent chance at good 1 alone for a price of 0.2 or the full bundle for sure for a price of 1.

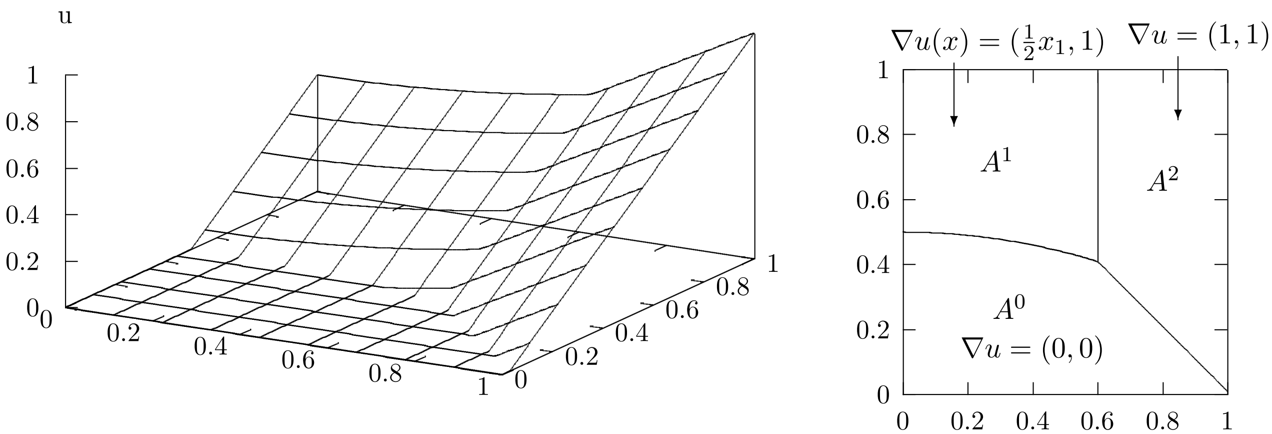

Figure 3: $u(x) = \max\{0,\ (0.25x_1^2 + x_2 - 0.5), (x_1 + x_2 - 1.01)\}$

Example 2 *A non-piecewise linear extreme point. Let $N = 2$ and let $u \in W$ be defined by*

$$u(x) = \max\{0, (0.25x_1^2 + x_2 - 0.5), (x_1 + x_2 - 1.01)\}.$$

The mechanism u is piecewise differentiable but not piecewise linear. The function u and its corresponding pieces are depicted in Figure 3.

The mechanism u, determines three pieces, $A^0 = \{x \in I^N : \nabla u(x) = (0,0)\}$, $A^1 = \{x \in I^N : \nabla u(x) = (\frac{1}{2}x_1, 1)\}$, and $A^2 = \{x \in I^N : \nabla u(x) = (1,1)\}$. The boundary between A^0 and A^1 is $A^0 \cap A^1 = \{x \in I^N : x_1 \in [0, 0.6],$ and $x_2 = \frac{1}{2} - \frac{1}{4}x_1^2\}$.

Suppose u is not an extreme point. Then there is a function $g(x)$ such that both $u+g$ and $u-g$ are in W. For any x in $A^0 \cup A^2$, $\nabla g(x)$ is $(0,0)$, and by continuity, $g(x)$ must also be 0. If $g \neq 0$, there must therefore be an element $x' = (x_1', x_2')$ in A^1 such that $g(x') \neq 0$. Suppose without loss of generality that $g(x') > 0$. (If $g(x') < 0$, a similar argument will apply to $u - g$.) Let $y' = \frac{1}{2} - \frac{1}{4}x_1'^2$; thus (x_1', y') is the point on the boundary between A^0 and A^1, directly below (x_1', x_2'). On that boundary both g and u must be zero; thus $g(x_1', y') = 0 = u(x_1', y')$. Because $u+g$ must be convex and its gradient is in $[0,1]^2$,

$$u(x_1', x_2') + g(x_1', x_2') \leq u(x_1', y') + g(x_1', y') + (x_1' - x_1') + (x_2' - y') = (x_2' - y).$$

Since $u(x_1', x_2') = u(x_1', y') + (x_2' - y')$, we obtain, $g(x_1', x_2') \leq g(x_1', x_2') = 0$, a contradiction.

Both examples illustrate that extreme points may include randomization in the assignment of goods to customers. Example 1 demonstrates that randomization can occur even with piecewise linear mechanisms. Example 2 demonstrates that an extreme point need not be piecewise linear. In both examples, randomization takes place on the assignment of a single good. Example 3 in Subsection 6.2 presents a piecewise linear extreme point with randomization over all goods.

5 Revenue Maximization

We have shown, by example, that extreme points in higher dimensions are essentially different from those found in one-dimensional environments. We have not shown, however, that those "different" extreme points are the solution to a relevant seller's problem. In this section we characterize the mechanisms that maximize seller's revenue for some prior distribution of buyer's valuations. The examples discussed so far are shown to be within this class.

There are extreme points of W that are never a best choice for the seller. Two such extreme points are the mechanism in which no buyer ever gets an object (i.e., $\nabla \bar{u} = \mathbf{0}$), and the mechanism in which buyers always get the object (i.e., $\nabla \bar{u} = \mathbf{1}$). That these mechanisms are extreme points follows easily from the definition noting that the vector of probabilities of trade, $\nabla u(x)$, equals $\mathbf{0}$ and $\mathbf{1}$ respectively. Both mechanisms, however, always yield zero revenue to the seller, $t = 0$. Clearly, the seller will not use the mechanisms described. There are alternative mechanisms, for instance the mechanism $u'(x) = \max\left\{0, (1 \cdot x - \frac{N}{2})\right\}$, which always yield at least as much revenue.

In order to identify mechanisms that are the solution to the seller's problem for some prior density of buyer's valuations, we restate the program so that the optimization variable is the transfer function t. We do so for two reasons. First, it is immediate to characterize in terms of the transfer functions the mechanisms that maximize seller's revenue for some prior density of valuations. Second, transfers underline the geometric quality of our arguments. Both points are developed throughout this section.

Definition 3 *The feasible set of transfer functions in the seller's problem is*

$$T = \{t \;:\; t(x) = \nabla u(x) \cdot x - u(x) \text{ a.e.}, u \in W\}.$$

Since feasible mechanisms $u \in W$ need only be differentiable almost everywhere in I^N, their corresponding transfers t are only defined for almost all x in I^N.

Remark 1 *It is simple to verify that T is convex, L^1-compact (Lemma 3), and that for any extreme point $\bar{u} \in W$, its corresponding transfer function $\bar{t}$ is an extreme point of T. For any $u \in W$ there is a $t \in T$. However, for some $t \in T$ there may be many $u \in W$ that generate it.*

In terms of transfers, the seller's problem is

$$\max_{t \in T} E[t]. \tag{2}$$

Although both forms of the seller's problem—program (1) in terms of payoffs u and program (2) in terms of transfers t—are equivalent, the latter has a more transparent geometric interpretation. The expectation in the objective function of both problems is taken with respect to a density of buyer's valuations, the seller's prior beliefs. If $f \in L^\infty(I^N)$ is such density function, then $E[t] = \int_{I^N} t(x) f(x)\, dx = \langle t, f \rangle$. The latter notation highlights the bilinear relationship between the density f and the transfer t; f may be seen as a linear function with t as argument, and t is a linear function with f as argument. For any real number r, the set $\{g \in L^1(I^N) : \langle g, f\rangle = r\}$ represents a "hyperplane" in the space L^1.

Intuitively, a mechanism is undominated if there is no alternative mechanism yielding always at least as much revenue to the seller and strictly more in some cases. (A formal definition is provided below.) We prove that, for any undominated mechanism $\bar{t} \in T$, there is a density over valuations f for which $\bar{t}$ is a revenue-maximizing mechanism. That is to say, $\langle \bar{t}, f\rangle \geq \langle t, f\rangle$ for all $t \in T$; or equivalently, there is a hyperplane supporting T at $\bar{t}$.

Definition 4 *A mechanism $t \in T$ is dominated if there is an alternative mechanism $t' \in T$ such that $t'(x) \geq t(x)$ a.e. in I^N, with strict inequality in a set of positive Lebesgue measure. A mechanism t is undominated if it is not undominated. (We will say a mechanism $u \in W$ is undominated if its corresponding transfer $t(x) = \nabla u(x) \cdot x - u(x)$ is undominated.)*

Definition 5 *An integrable function $f : I^N \longrightarrow \mathbb{R}_+$ is a density function if $\int_{I^N} f(x)dx = 1$. In addition f satisfies independence if $f(x) = f_1(x_1) \times \ldots \times f_N(x_N)$, where for $i = 1, \ldots N$, $f_i(x_i) = \int f(x_i, x_{-i})\, dx_{-i}$.*

The following theorem shows that any mechanism $\bar{t}$ which is undominated is optimal for some seller beliefs. Furthermore, the result holds even if we restrict attention to the narrower class of densities where the buyer's valuations for each good are distributed independently. We briefly describe its proof; the same approach may apply to other classes of prior densities. First, the set $\mathcal{F}$ from which the supporting density will be obtained is defined. (In our case the set of essentially bounded densities satisfying independence.) For each $f \in \mathcal{F}$ there is a mechanism t_f that yields higher expected revenue than the proposed $\bar{t}$. (Otherwise the claim is established.) Any density sufficiently close to f will also yield a higher expected revenue under t_f than under t. Compactness of $\mathcal{F}$ implies that we can select finitely many mechanisms $\{t_f\}$, so that under any density, one of those t_f will give higher revenue than $\bar{t}$. A convex combination $\tilde{t}$ of those finitely many transfers $\{t_f\}$ is constructed using a finite-dimensional separating hyperplane argument to obtain the weights. It is shown that for any density in $\mathcal{F}$, $\tilde{t}$ yields higher expected revenue than $\bar{t}$. Only the compactness of $\mathcal{F}$ has been used so far. To prove that $\tilde{t}$ dominates $\bar{t}$, the set $\mathcal{F}$ must be sufficiently rich. Let E be the set of buyer types where $\bar{t}(x) > \tilde{t}(x)$. The set of possible densities $\mathcal{F}$ must include some density with support in E. Then E must have zero measure or the separation established earlier would be violated. In summary, since $\mathcal{F}$ is weak* compact, and it includes sufficiently many densities, the argument holds.

Theorem 3 *Let $\bar{t} \in T$ be undominated. Then there is a density function $f \in L^\infty$ satisfying independence for which $\bar{t}$ maximizes expected revenue.*

Proof Let $\mathcal{F}$ be the set of independent density functions $f \in L^\infty(I^N)$. The set $\mathcal{F}$ is weak* compact.

For each $f \in \mathcal{F}$, select $t_f \in T$ such that $\langle t_f, f\rangle > \langle \bar{t}, f\rangle$. If, for some $f \in \mathcal{F}$, no such t_f exists, then for that f, $\langle \bar{t}, f\rangle \geq \langle t, f\rangle\, \forall t \in T$ and the proof is complete.

By continuity, there is a weak* open neighborhood $O_f \ni f$ such that

$$f' \in O_f \implies \langle t_f, f'\rangle > \langle \bar{t}, f'\rangle. \tag{3}$$

The collection $\{O_f : f \in \mathcal{F}\}$ is an open cover of $\mathcal{F}$; by compactness it has a finite subcover $\{O_m : m = 1, \ldots, M\}$. Denote by $\{t_1, t_2, \ldots, t_M\}$ the corresponding transfer functions identified in (3). The identified transfer functions are now used to construct a weakly dominant strategy t' using a finite-dimensional separating-hyperplane argument.

Let

$$G = \{\langle t_1 - \bar{t}, f\rangle, \langle t_2 - \bar{t}, f\rangle, \ldots, \langle t_M - \bar{t}, f\rangle : f \in \mathcal{F}\}$$

The set G is a convex subset of $\mathbb{R}^M$ and $G \cap \mathbb{R}^M_- = \emptyset$. Therefore, there is a separating hyperplane $\alpha \in \mathbb{R}^M_+$ such that $\alpha \cdot y > 0\ \forall y \in G$. Without loss of generality, we may normalize α so that $\sum_{i=1}^M \alpha_i = 1$ Let

$$\tilde{t} = \alpha \cdot (t_1, \ldots t_M).$$

Since T is convex, $\tilde{t} \in T$. Observe that

$$\forall f \in \mathcal{F}\ \langle \tilde{t}, f\rangle - \langle \bar{t}, f\rangle = \alpha \cdot (\langle t_1 - \bar{t}, f\rangle, \ldots, \langle t_M - \bar{t}, f\rangle) > 0. \tag{4}$$

Since f is arbitrary within $\mathcal{F}$, it must be the case that $\tilde{t}$ dominates $\bar{t}$. To see this, let $E = \{x \in I^N : \bar{t}(x) > \tilde{t}(x)\}$. This set is measurable. Suppose $\lambda(E) > 0$. Let $\mathcal{D} = \{A \subset I^N : \langle \tilde{t}, 1_A\rangle \geq \langle \bar{t}, 1_A\rangle\}$. Note that $\mathcal{D}$ is a $\pi - class$, and a $\lambda - class$. Then $\mathcal{D}$ is a sigma field. Since f comes from the class of independent densities, (4) implies that all measurable rectangles in I^N are in $\mathcal{D}$ and, therefore, $\mathcal{D}$ must include the Borel sigma field in I^N. Thus $E \in \mathcal{D}$. This proves that $\tilde{t} \geq \bar{t}$ a.e. in I^N. If the two functions were equal, the separation in (4) would not be strict. *Q.E.D.*

Note also that Theorem 3 applies to *every* undominated t in T, not just the extreme points.

Remark 2 *The supporting density function identified in Theorem 3 need not have full support in I^N.*

Theorem 4 below presents a property of undominated mechanisms that links domination, defined on transfers, with the behavior of the corresponding payoff functions. We use this property to show, among other things, that the extreme points in our Examples are undominated. According to the theorem, if a mechanism $t_{u'}$ dominates a mechanism t_u, and if $u'(x)$ exceeds $u(x)$ for some x, then u' must remain above u for all points farther out along the ray through the origin containing x.

Theorem 4 *Let u and u' be two mechanisms in W and let t and t' denote their corresponding transfer functions. Suppose t' dominates t and let x be any element of I^N. Then,* *1.* $u'(x) > u(x) \implies u'(\delta x) > u(\delta x)$ *for all* $\delta x \in I^N$, $\delta > 1$, *and*

2. $u'(x) \geq u(x) \implies u'(\delta x) \geq u'(\delta x)$ *for all* $\delta x \in I^N$, $\delta > 1$.

Proof Part 1. Let $u'(x) > u(x)$ and suppose that for some $\delta > 1$, $u'(\delta x) \leq u(\delta x)$. Let $\delta' = \inf\{\delta > 1 : u'(\delta x) \leq u(\delta x)\}$. By continuity, $u'(\delta' x) - u(\delta' x) = 0$ and $\delta' > 1$. Furthermore, $u'(\delta x) > u(\delta x)$ for all $\delta \in (1, \delta')$.

By definition, $t' = \nabla u' \cdot x - u'$ and $t = \nabla u \cdot x - u$ almost everywhere. Since t' dominates t, $(\nabla u'(x) - \nabla u(x)) \cdot x - (u'(x) - u(x)) \geq 0$ for almost all $x \in I^N$.

We will prove the theorem under two additional assumptions and show afterwords that the two assumptions are always satisfied. Suppose for the moment that

$$\forall \delta \in (1, \delta'), \;\; -[u'(\delta x) - u(\delta x)] = \int_{\delta}^{\delta'} [\nabla u'(\gamma x) - \nabla u(\gamma x)] \cdot x \, d\gamma, \text{ and that} \tag{5}$$

$$\nabla u'(x) \cdot x - u'(x) \geq \nabla u(x) \cdot x - u(x), \; \forall x \in I^N. \tag{6}$$

(Note that (5) holds immediately if u and u' are differentiable everywhere. Assumption (6) simply states that $t'(x) \geq t(x)$ everywhere in I^N. When u and u' are differentiable, t and t' are defined everywhere and (6) holds, provided t' dominates t.)

Using (1) and our observation that $u'(\delta x) - u(\delta x) > 0$ for $\delta \in (1, \delta')$, we obtain

$$\forall \delta \in (1, \delta'), \;\; -[u'(\delta x) - u(\delta x)] = \int_{\delta}^{\delta'} [\nabla u'(\gamma x) - \nabla u(\gamma x)] \cdot x \, d\gamma < 0.$$

From (2), it follows in particular, that for all γ in (δ, δ'), we have that $(\nabla u'(\gamma x) - \nabla u(\gamma x)) \cdot \gamma x \geq u'(\gamma x) - u(\gamma x) > 0$. This implies that $[\nabla u'(\gamma x) - \nabla u(\gamma x)] \cdot x > 0$, which contradicts (5) and proves Part 1 under our two additional assumptions.

That the two extra assumptions are unnecessary follows from Lemma 6 in the Appendix. There we construct selections from the subdifferential of u and u' satisfying both assumptions.

Part 2. A similar argument to that used in Part 1 suffices; we sketch it in the following lines. Suppose in this case that $u'(x) < u(x)$ and for some $\delta < 1$, $u'(\delta x) \geq u(\delta x)$. Let $\delta' = \sup\{\delta < 1 : u'(\delta x) \geq u(\delta x)\}$. By continuity, $u'(\delta' x) - u(\delta' x) = 0$ and $\delta' < 1$. Furthermore, $u'(\delta x) < u(\delta x)$ for all $\delta \in (\delta', 1)$. The proof continues as in Part 1. *Q.E.D.*

A corollary illustrates the usefulness of Theorem 4 in identifying undominated mechanisms. It also highlights that, depending on priors, the type of revenue-maximizing mechanism varies significantly.

Corollary 1 *The mechanisms described in Examples 1 and 2 are undominated, and hence they maximize expected seller's revenue for some prior density of buyer's valuations.*

Proof Before considering each example individually, we highlight the following consequence of Theorem 4 for later use:

Let u and $\bar{u}$ be feasible mechanisms such that the transfer function, t, associated with u dominates $\bar{t}$, the transfer function associated with $\bar{u}$. Then

$$[u \geq \bar{u} \text{ and for some } x \in I^N, u(x) = \bar{u}(x)] \implies u(\delta x) = \bar{u}(\delta x) \; \forall \delta \in [0,1]. \tag{7}$$

We now consider the examples individually. In both examples we suppose, arguing by contradiction, that there is a mechanism $u \in W$ with transfer t, and that t dominates $\bar{t}$, the transfer associated with $\bar{u} \in W$. In each example $\bar{u}$ represents the candidate extreme point.

Example 1. It is useful to revisit Figure 2.

First, we establish that $u(x) \geq \bar{u}(x)$ for all x. Note that if $x \in A^0$, $u(x) \geq \bar{u}(x) = 0$. Theorem 4 then implies that $u(y) \geq \bar{u}(y)$ for all y in $\mathcal{R}_x \cap I^N$. The set I^N is a subset of $\bigcup_{x \in A^0} \mathcal{R}_x$.

Second, we show that $u(x) = \bar{u}(x)$ for any $x \in A^2$. Since t dominates $\bar{t}$, we have $t(x) \geq \bar{t}(x)$, or equivalently $\nabla u(x) \cdot x - u(x) \geq \nabla \bar{u}(x) \cdot x - \bar{u}(x)$ a.e. in I^N. We also have $\nabla \bar{u}(x) = \mathbf{1}$ a.e. in A^2. Therefore $0 \geq (\nabla u(x) - \mathbf{1}) \cdot x \geq u(x) - \bar{u}(x) \geq 0$ a.e. in A^2. It follows that $u(x) = \bar{u}(x)$ a.e in A^2. Continuity implies the desired result.

Third, we prove that $u(y) = \bar{u}(y)$ for any $y \in \left[I^N \cap \left(\bigcup_{x \in A^2} \mathcal{R}_x\right)\right]$. This follows from (7).

Fourth, we show that $u(y) = \bar{u}(y)$, for any $y \in \left[I^N \setminus \left(\bigcup_{x \in A^2} \mathcal{R}_x\right)\right]$. Pick any such $y = (y_1, y_2)$ and suppose by way of contradiction that $u(y) > \bar{u}(y)$. It must be the case that $y_2 < 0.3y_1$; otherwise y would belong to $\left[I^N \cap \left(\bigcup_{x \in A^2} \mathcal{R}_x\right)\right]$. Let $y_2' = 0.3y_1$. Then, $u(y) > \bar{u}(y) = \bar{u}(y_1, y_2') = u(y_1, y_2')$. Hence u is not monotone. It follows that $u \notin W$, a contradiction.

We have demonstrated that $u = \bar{u}$. This implies $t = \bar{t}$ and therefore t does not dominate $\bar{t}$.

Example 2. Figure 4 is useful in following the proof.

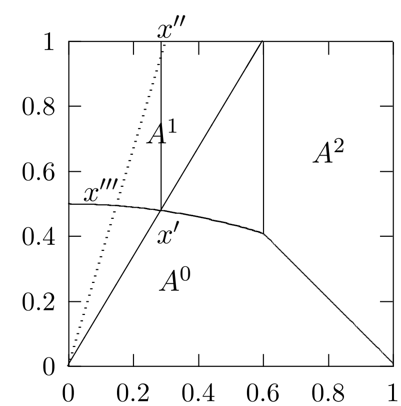

Figure 4: Proof of Corollary 1

Similar arguments to those used in the previous example establish that first, $u \geq \bar{u}$; second, $u(x) = \bar{u}(x)$ for all $x \in A^2$; and third, $u(y) = \bar{u}(y)$ for all $y \in \left[I^N \cap \left(\bigcup_{x \in A^2} \mathcal{R}_x\right)\right]$. Note that the area underneath the solid line in the figure, $I^N \cap \left(\bigcup_{x \in A^2} \mathcal{R}_x\right)$, is $\{(y_1, y_2) \in I^N : \frac{3}{5} y_2 \leq y_1\}$.

Fourth, let $x' = (x_1', x_2')$ be the intersection of the line $x_2 = \frac{5}{3} x_1$, and the line defining the boundary between A^0 and A^1, $x_2 = \frac{1}{2} - \frac{1}{4} x_1^2$. We show that $u(y) = \bar{u}(y)$ for any y with $y_1 \in [x_1', 0.6]$

and $y_2 > \frac{5}{3}y_1$. Pick any such $y = (y_1, y_2)$, and suppose $u(y) > \bar{u}(y)$. Note that $(y_1, \frac{5}{3}y_1) \in \left[I^N \cap \left(\bigcup_{x \in A^2} \mathcal{R}_x\right)\right]$, and therefore $u(y_1, \frac{5}{3}y_1) = \bar{u}(y_1, \frac{5}{3}y_1)$. Using the fact that $0 < \nabla u_2 < 1$, that $\nabla \bar{u}_2 = 1$, and that $u(y) > \bar{u}(y)$ we generate the following contradiction, $\left(y_2 - \frac{5}{3}y_1\right) + u(y_1, \frac{5}{3}y_1) \geq u(y) > \bar{u}(y) = \bar{u}(y_1, \frac{5}{3}y_1) + \left(y_2 - \frac{5}{3}y_1\right)$. We have proved that $u'(y) = u(y)$ for any $y \in E$, where $E \subset I^N$ is the union of the convex hull of $\{(0,0), x', (1,0)\}$ and the convex hull of $\{x', x'', (1,1), (1,0)\}$.

Fifth, we prove that $u'(y) = u(y)$ for any $y \in (I^N \cap \bigcup_{x \in E} \mathcal{R}_x)$. This follows from (7).

Note that the proof proceeds by showing that $u = \bar{u}$ in a given area, and then that in the rays defined by that area u must also equal $\bar{u}$. We continue with this procedure. The intersection of the segment $[\mathbf{0}, x'']$ with the boundary $A^0 \cap A^1$, defines a point x'''. In turn the points $(x''', (x_1''', 1), x'', x')$ define a new area. Arguments similar to those used in point four, establish that $u = \bar{u}$ in that region. Continuing with this process establishes that $u = \bar{u}$ for all $x \in I^N$. *Q.E.D.*

We conclude this section introducing a stronger notion of domination, sequential domination. We prove that sequentially dominated mechanisms never maximize revenue if the prior density has full support, more precisely if the prior density is bounded away from zero.

Recall that given any real-valued function g defined on I^N, $g_+(x)$ is the maximum of $\{g(x), 0\}$ and $g_-(x)$ is the maximum of $\{-g(x), 0\}$.

Definition 6 *A mechanism $t \in T$ is sequentially dominated if for $n = 1, 2, \ldots$ there is t^n in T, $t^n \neq t$ such that $t^n \xrightarrow{L_1} t$, and*

$$\frac{\|(t^n - t)_+\|_1}{\|(t^n - t)_-\|_1} \longrightarrow \infty.$$

A mechanism t is sequentially undominated if it is not sequentially dominated.

Given a mechanism t, the definition above requires that there be a sequence of mechanisms $\{t^n\}$ approximating t. The gains from using the mechanism t^n instead of t are given by $\|(t^n - t)_+\|_1$. The losses are given by $\|(t^n - t)_-\|_1$. If t is undominated, any mechanism t^n cannot yield only gains when compared to t. Since the sequence $\{t^n\}$ approximates t, both gains and losses must vanish as n goes to infinity. That is, both the numerator and the denominator in the definition go to zero. The mechanism t is sequentially dominated, if the losses go to zero significantly faster than the gains. Thus, far along the sequence, the gains overcome the losses.

Remark 3 *If t is dominated, then it is sequentially dominated.*

Theorem 5 *Let $f \in L^\infty$ be any prior density function such that for some $\epsilon > 0$, $f(x) > \epsilon \, \forall x \in I^N$. If $\bar{t} \in T$ is sequentially dominated, then $\bar{t}$ does not maximize expected seller's revenue.*

Proof Let $t^n \in T$, $n = 1, 2, \ldots$, be a sequence dominating $\bar{t}$. Suppose that $\langle f, \bar{t} \rangle \geq \langle f, t \rangle \, \forall t \in T$. Then, $0 \geq \langle f, t^n - \bar{t} \rangle = \langle f, (t^n - \bar{t})_+ \rangle - \langle f, (t^n - \bar{t})_- \rangle, \forall n$. Thus, $\langle f, (t^n - \bar{t})_- \rangle \geq \langle f, (t^n - \bar{t})_+ \rangle \, \forall n$. Then for all n,

$$\|f\|_\infty \|(t^n - \bar{t})_-\|_1 \geq \langle f, (t^n - \bar{t})_- \rangle \geq \langle f, (t^n - \bar{t})_+ \rangle \geq \epsilon \|(t^n - \bar{t})_+\|_1.$$

This implies that

$$\frac{\|f\|_\infty}{\epsilon} \geq \frac{\|(t^n - \bar{t})_+\|_1}{\|(t^n - \bar{t})_-\|_1}.$$

Since t^n sequentially dominates $\bar{t}$, the right-hand side of the expression above goes to infinity, a contradiction. *Q.E.D.*

Remark 4 *For any $\epsilon > 0$ and any density f, with $f(x) > \epsilon$ for all $x \in I^N$, there is a sequentially undominated extreme point of the feasible set of mechanisms that maximizes expected seller's revenue. (This is a consequence of Weierstrass's Theorem and the Bauer Maximum Principle.) In particular, this shows the existence of sequentially undominated extreme points.*

6 The Structure of Potential Solutions

We explore in this Section the structure of the feasible set. Recall that for any prior density of valuations, the set of maximizers in the seller's problem is a face of the feasible set W (Lemma 4). We characterize a class of relevant faces. Based on this characterization, we advance a procedure to determine whether a proposed mechanism is an extreme point.

Before presenting our findings, we summarize them, although not in the order in which they are derived. First, we show that restricting attention to piecewise linear mechanisms is, essentially, without loss of generality. Non-piecewise linear mechanisms may be extreme points (Example 2), and they may even maximize expected revenue (Corollary 1). That piecewise linear mechanisms are dense in W is a straightforward observation (Lemma 5). We demonstrate that, in addition, the set of *piecewise linear extreme points*, is dense in the set of *all* extreme points (Theorem 10). Since expected seller's revenue is always maximized at an extreme point (Bauer Maximum Principle), there is little loss in restricting attention to *piecewise linear* extreme points.

Second, we show that it is comparatively simple to verify whether a *piecewise linear* mechanism is an extreme point. Generally, a mechanism $\bar{u}$ in W is an extreme point when it is *not* possible to move from $\bar{u}$ in any direction g and in the opposite direction $-g$, remaining in both cases within the feasible set W, i.e. $\bar{u} + g$ or $\bar{u} - g$ must be outside W.

Thus, to determine whether a given mechanism is an extreme point, the number of directions g to check is quite large. The situation is simpler, however, when $\bar{u}$ is piecewise linear. Piecewise linear mechanisms partition the set of buyers in finitely many pieces or subsets such that consumer types in each piece are treated similarly by the mechanism. We demonstrate that to verify whether a piecewise linear mechanism $\bar{u}$ is an extreme point, it suffices to check directions g that are also piecewise linear, and that define *the same pieces* or subsets as $\bar{u}$ does (Theorem 7). This observation is fundamental in the sense that all other results in the section rely on it.

Third, we characterize some important faces of W. Pick any piecewise linear mechanism $\bar{u}$ and its implicit partition of buyer's types. We define a face relative to $\bar{u}$ and more importantly, relative to the partition defined by $\bar{u}$. More precisely, the collection of all piecewise linear mechanisms with coarser partitions of buyer's types than the partition defined by $\bar{u}$ is a face $F_{\bar{u}}$ of W (Theorem 8).

Fourth, we describe how our characterization of faces is the basis for an algebraic method to identify extreme points. Determining whether a piecewise linear mechanism is an extreme point is, essentially, equivalent to determining if a consistent, linear system of equations has a unique solution.

We present our result in three subsections. The first contains the main results of the section. The second consists of an example of an undominated extreme point that involves randomization for all goods. The example also illustrates the use of some of the results developed in the first subsection. The last subsection describes how to use our characterization of faces to determine if a piecewise linear mechanism is an extreme point. We use the example to illustrate the methodology.

6.1 Piecewise Linear Mechanisms

A function u is piecewise linear if it consists of finitely many linear pieces. Because of incentive compatibility, feasible mechanisms are the pointwise supremum of linear functions with gradient in the N-dimensional unit cube (see Theorem 1 and the discussion surrounding it). Because of individual rationality, one of those linear functions is the null map. A piecewise linear mechanism must, therefore, be the pointwise maximum of *finitely* many linear functions. These observations establish the following remark.

Remark 5 *The mechanism u is piecewise linear and feasible if and only if there is a finite family of linear functions, $u^j(x) = a^j \cdot x - b^j$ with $a^j \in I^N$ and $b^j \in \mathbb{R}$ for $j = 0, 1, \ldots, J$, such that for every x in I^N, $u(x) = \max\{u^j(x) : j = 0, 1, \ldots, J, u^0 = 0\}$.*

A piecewise linear mechanism partitions the set I^N of consumer types into finitely many groups. Types within each group are treated equally, in the sense that they all face the same probabilities of trade and pay the same transfer. We refer to those groups as market segments. Market segments are the effective domains of the different linear pieces forming the mechanism.

Definition 7 *Let u be a piecewise linear mechanism in W, and let $\{u^j\}_{j=0}^J$ be the smallest (by set inclusion) family of linear functions such that $u(x) = \max\{u^j(x) : j = 0, 1 \ldots, J\}$. For each $j = 0, 1, \ldots, J$, we say that $A^j = \{x \in I^N : u^j(x) > u^k(x) \, \forall k \neq j\}$ is a market segment of the mechanism u. We denote by $m(u)$ the collection of all such market segments.*

Let A^j be in $m(u)$. We denote by ∇u^j the gradient of u in A^j (i.e., $\nabla u^j = \nabla u(x)$ for every x in A^j).

Let t be the transfer function associated with u. We denote by t^j the transfer from members of A^j to the seller (i.e., $t^j = t(x)$ for every $x \in A^j$).

A market segment is a collection of buyer types x satisfying finitely many, linear, strict inequalities. Redundant pieces, such as those that are never a maximum or those that are, at best, a weak maximum, are eliminated from the definition. From this considerations we derive the following remark.

Remark 6 *Given a piecewise linear, feasible mechanism u, its market segments are convex, and relatively open subsets of I^N with full dimension. Given any two market segments A^j and A^k, $k \neq j$, then $\nabla u^j \neq \nabla u^k$.*

The following Theorem states that any undominated, piecewise-linear mechanism must include a market segment where all goods are traded with certainty, and a market segment where there is no trade at all.

Theorem 6 *Let u be an undominated, piecewise linear mechanism in W. Then there are market segments A^0 and A^J such that no good is assigned if the buyer's type is in A^0, and all goods are assigned with certainty if the buyer's type is in A^J, i.e. $\nabla u^0 = \mathbf{0}$ and $\nabla u^J = \mathbf{1}$.*

We relegate the proof to the Appendix.

Armstrong (1996) shows that when there are at least two objects and the support of the prior density of buyer's valuations is strictly convex to the origin, the optimal mechanism will assign no goods to some group of buyers. Theorem 6 is not implied by (nor it implies) Armstrong's exclusionary principle. Theorem 6 describes the structure of the feasible set; it does not depend on the prior density. The no-trade region in our theorem contains the origin and since the prior density of buyer's valuations need not have full support, the excluded buyers may be negligible with respect to the prior density of valuations.

Theorem 7 is the fundamental building block of this section. To determine that a mechanism u is *not* an extreme point of W it suffices to find a single function g such that moving from u in the direction g yields a feasible mechanism $u + g \in W$, and moving in the opposite direction also yields a feasible mechanism $u - g \in W$. To determine that a mechanism $\bar{u}$ *is* an extreme point, however, involves verifying that $u + g$ or $u - g$ are not feasible for every possible direction g. Theorem 7 reduces significantly the number of directions that must be verified when dealing with piecewise linear mechanisms. It states that if $\bar{u}$ is piecewise linear, it suffices to verify only the piecewise linear, continuous functions g whose pieces have, as effective domain, the market segments of $\bar{u}$.

Theorem 7 *Let $\bar{u}$ be a piecewise linear mechanism. The mechanism $\bar{u}$ is an extreme point of W if and only if $\bar{u} + g \notin W$ or $\bar{u} - g \notin W$, for every continuous, piecewise linear function $g : I^N \longrightarrow \mathbb{R}$ such that $A^j \in m(\bar{u})$ implies g is linear on A^j.*

Proof By definition of extreme point, necessity is obvious.

We prove sufficiency. If $\bar{u}$ is not an extreme point of W, then there is a function g such that $\bar{u} + g \in W$ and $\bar{u} - g \in W$. This implies that g must be continuous for otherwise $\bar{u} + g$ is not continuous.

Pick any market segment A^j in $m(\bar{u})$. The restriction of u to A^j is linear. Both $\mathtt{1}_{A^j}(\bar{u} + g)$ and $\mathtt{1}_{A^j}(\bar{u} - g)$ are convex when restricted to the domain A^j. Therefore, $\mathtt{1}_{A^j} g$ must be linear within A^j. *Q.E.D.*

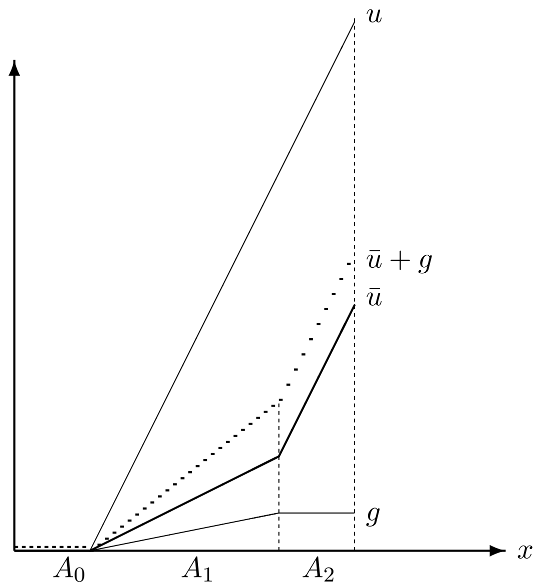

Figure 5: Identifying Extreme Points

Figure 5 illustrates Theorem 7. The mechanism $\bar{u}$ determines three market segments, A^0, A^1, A^2. To determine if $\bar{u}$ is an extreme point of W, it suffices to check whether $\bar{u}+g$ and $\bar{u}-g$ are in W for functions g that are linear on the market segments of $\bar{u}$.

We use Theorem 7 repeatedly in this and the next Sections. We also use it to show that the mechanism in Example 3 is an extreme point. In that example *all goods* are assigned randomly within a market segment.

We now characterize some very useful faces of W. Theorem 8 states, roughly, that given a piecewise linear mechanism $\bar{u}$, the set of piecewise linear mechanisms in W with the same market segments as $\bar{u}$ is a face of W.

Theorem 8 *Let $\bar{u}$ be any piecewise linear mechanism in W. Define the set*

$$F_{\bar{u}} = \{u \in W : \forall A \in m(\bar{u}),\ u \text{ is linear on } A;\ \text{and } [\nabla_i \bar{u}(x) \in \{0,1\} \implies \nabla_i u(x) = \nabla_i \bar{u}(x)]\}.$$

The set $F_{\bar{u}}$ is a face of W.

Proof Let u be any element of $F_{\bar{u}}$. Suppose $u = 1/2u' + 1/2u''$, for some $u', u'' \in W$, $u' \neq u''$. Pick any A in $m(u)$, and suppose u' is not linear on A. Then, since u' is convex, there are $x', x'' \in A$ such that $u'(\bar{x}) < \frac{u'(x')+u'(x'')}{2}$ where $\bar{x} = \frac{x'+x''}{2}$. Note that

$$\begin{aligned} u''(\bar{x}) - \frac{1}{2}[u''(x') + u''(x'')] &= 2u(\bar{x}) - u'(\bar{x}) - [u(x') - \frac{1}{2}u'(x')] - [u(x'') - \frac{1}{2}u'(x'')] \\ &= [2u(\bar{x}) - u(x') - u(x'')] - [u'(\bar{x}) - \frac{1}{2}u'(x') - \frac{1}{2}u'(x'')] \\ &= -[u'(\bar{x}) - \frac{1}{2}u'(x') - \frac{1}{2}u'(x'')] > 0, \end{aligned}$$

which implies that u'' is not convex and therefore u'' is not an element of W, a contradiction. We conclude that u' must be linear on A^j. A symmetric argument shows that u'' must also be linear on A. Since A is arbitrary, both u' and u'' must be linear on each A. This proves that $F_{\bar{u}}$ is an extreme set of W. Noting that $F_{\bar{u}}$ is also convex, completes the proof. *Q.E.D.*

The face $F_{\bar{u}}$ is defined as the set of all piecewise linear, feasible mechanisms u, that are linear on every market segment A of $\bar{u}$ and that satisfy an additional restriction on gradients. Consumers in two different market segments of $\bar{u}$, may be treated equally by some u in $F_{\bar{u}}$; for instance, in Figure 5, consumers in A^1 and A^2 in $m(\bar{u})$ are treated equally by u.

The definition of $F_{\bar{u}}$ includes a gradient restriction: If under the proposed mechanism $\bar{u}$ the probability $\nabla_i \bar{u}^j$ of assigning object i to a consumer in market segment A^j is either zero or one, then to be an element of $F_{\bar{u}}$, any alternative mechanism u must also assigned good i with the same probability $\nabla_i u^j = \nabla_i \bar{u}^j$ (zero or one) to any consumer in market segment A^j.

If we did not impose the discussed restriction on gradients, the resulting set of mechanisms would still be a face of W. However, not all faces of W are useful for our problem. For instance, the entire set W is a face, and the singleton containing any extreme point is a face. The faces we defined are useful to identify extreme points. Pick any piecewise linear mechanism $\bar{u}$ and consider the face $F_{\bar{u}}$ described earlier. Theorem 9 below demonstrates that $\bar{u}$ is an extreme point if and only if $F_{\bar{u}}$ is the singleton $\{\bar{u}\}$.

Theorem 9 *Let $\bar{u}$ be a piecewise linear element of W. Then $\bar{u}$ is an extreme point of W if and only if $F_{\bar{u}} = \{\bar{u}\}$.*

Proof One direction is trivial: Theorem 8 demonstrates that $F_{\bar{u}}$ is a face of W; therefore if $F_{\bar{u}} = \{\bar{u}\}$, $\bar{u}$ is an extreme point of W.

We prove the converse. Suppose $\bar{u}$ is a piecewise linear element of W and suppose there is $u' \in F_{\bar{u}}$, $u' \neq \bar{u}$. We will show that $\bar{u}$ is not an extreme point.

Let $\{A^j\}_{j=0}^J$ be the family of all market segments $m(\bar{u})$. It follows from the definition of $F_{\bar{u}}$, that both $\nabla\bar{u}$ and $\nabla u'(x)$ are constant in any market segment A^j in $m(\bar{u})$. For simplicity, we denote those constants as $\nabla\bar{u}^j$ and $\nabla u'^j$ respectively.

For $r \in [0, 1]$, define functions mapping I^N into $\mathbb{R}$ by by

$$\begin{aligned} v_r &= (1-r)\bar{u} + ru', \\ w_r &= (1-r)\bar{u} + r[2\bar{u} - u']. \end{aligned}$$

The functions v_r and w_r are piecewise linear, indeed they are linear on each market segment A^j in $m(\bar{u})$. For any such $A^j \in m(\bar{u})$, denote by ∇v_r^j and ∇w_r^j the gradients of v_r and w_r respectively (evaluated at any $x \in A^j$), and denote by $t_{v_r}^j$ and $t_{w_r}^j$ the corresponding intercepts. For any r, both ∇v_r and ∇w_r take at most $J+1$ values, the number of market segments defined by $\bar{u}$.

Pick any r. The function v_r is in W because it is the convex combination of elements of W. By construction $\bar{u}$ is the midpoint of the interval $[w_r, v_r]$. Hence, it suffices to show that for some

$r \in (0, 1)$, the function w_r is in W to prove that $\bar{u}$ is not an extreme point. We must prove that (i) $w_r(0) = \mathbf{0}$; (ii) ∇w_r is in I^N; and (iii) w_r is convex. Point (i) is obvious from the definition of w_r.

The proofs of (ii) and (iii) follow from an observation: w_r is piecewise linear, defines the same pieces as $\bar{u}$, and converges uniformly to $\bar{u}$. Since the gradient ∇w_r takes only finitely many values, ∇w_r also converges uniformly to the gradient $\nabla \bar{u}$. The details are below.

We verify (ii). We prove that there is an $r' \in (0, 1)$ such that for every $r \in (0, r')$ and every j, ∇w_r^j is in I^N.

If for some good i and market segment $A^j \in m(\bar{u})$, $\nabla_i \bar{u}^j \in \{0, 1\}$, then $\nabla_i u'^j = \nabla_i \bar{u}^j$. Thus, $\nabla_i w_r^j$ is in $\{0, 1\}$ for any r.

If for some good i and market segment j, $0 < \nabla_i \bar{u}^j < 1$, let

$$\epsilon = \min_{i,j} \left\{ \min\{(1 - \nabla_i \bar{u}^j), \nabla_i \bar{u}^j\} : 0 < \nabla_i \bar{u}^j < 1 \right\}.$$

The minimum is taken over finitely many values. As r tends to zero, the functions w_r converge uniformly to u. It follows from Lemma 3 that ∇w_r converges pointwise to $\nabla \bar{u}$. Hence there is $r_i^j \in (0, 1)$ such that $|\nabla_i w_r^j - \nabla_i \bar{u}^j| < \epsilon$ and therefore, $0 < \nabla_i w_r^j < 1$. Letting $r' = \min_{j,i}\{r_i^j\}$, the claim (ii) is established.

We verify (iii). We will prove that there is $r'' \in (0, 1)$ such that $r \in [0, r'')$ implies that w_r is convex.

For $x \in \mathbb{R}^N$, denote by

$$w_r^j(x) = \nabla w_r^j \cdot x - t_{w_r}^j.$$

The function w_r^j is the extension to the entire space $\mathbb{R}^N$ of the linear piece forming w_r on A^j. Similarly, we denote by $\bar{u}^j$ and u'^j the extensions of the linear pieces forming $\bar{u}$ and u' on A^j respectively. Thus, $w_r^j = (1 + r)\bar{u}^j - ru'^j$.

Fix any A^j and A^k in $m(\bar{u})$ such that $\dim(\bar{A}^j \cap \bar{A}^k) = N - 1$. For any $x \in A^j$,

$$w_r^j(x) - w_r^k(x) = (1 + r)[\bar{u}^j(x) - \bar{u}^k(x)] - r[u'^j(x) - u'^k(x)]. \tag{8}$$

Since A^j and A^k share an $(N - 1)$-dimensional boundary and since u' is linear on A^j and A^k, we obtain that

$$\exists \alpha \in \mathbb{R} : u'^j - u'^k = \alpha[\bar{u}^j - \bar{u}^k].$$

(This follows because $(u'^j - u'^k)$ and $(\bar{u}^j - \bar{u}^k)$ are affine operators with the exact same kernel of dimension $N - 1$.) Replacing the last expression in (8), we obtain that for any $x \in A^j$

$$w_r^j(x) - w_r^k(x) = (1 + r - r\alpha)[\bar{u}^j(x) - \bar{u}^k(x)].$$

The second factor is non-negative because $\bar{u}$ is convex and therefore the maximum of the linear functions forming it (Remark 5). There is $r_{j,k}$ in $(0, 1]$ such that for each $r \in [0, r_{j,k}]$ the first factor is strictly positive thus making the entire expression non-negative.

The value $r_{j,k}$ depends on the chosen market segments $A^j, A^k \in m(\bar{u})$. Let $r'' = \min\{r_{j,k} : A^j, A^k \in m(\bar{u}), \dim(\bar{A}^j \cap \bar{A}^k) = N - 1\}$. Since there are finitely many market segments, $r'' > 0$.

Hence we have proved that for every $r \in [0, r'']$, for every market segments $A^j, A^k \in m(\bar{u})$ with $\dim(\bar{A^j} \cap \bar{A^k}) = N - 1$, and for every $x \in A^j \cup A^k$,

$$w_r(x) = \max\{w_r^j(x), w_r^k(x)\}.$$

We now prove that w_r is convex. For any $(x, y) \in I^N \times I^N$, let

$$f(x, y) = \frac{1}{2}[w_r(x) + w_r(y)] - w_r\left(\frac{x + y}{2}\right).$$

Suppose by way of contradiction that w_r is not convex. Then there are market segments A^j, A^k in $m(\bar{u})$ and points $x' \in A^j$, $y' \in A^k$ such that $f(x', y') < 0$. Since f is continuous, there is an $\epsilon > 0$ such that for any $x \in B(x', \epsilon)$ and $y \in B(y', \epsilon)$, the function $f(x, y) < 0$.

Denote by $[x, y] = \{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\}$. Let $C = \{[x, y] : x \in B(x', \epsilon), y \in B(y', \epsilon)\}$. Then C is an N-dimensional cylinder.

There is $[x, y]$ in C such that any element $z \in [x, y]$ belongs to the closure of, at most, two market segments: if $z \in \overline{D'} \cap \overline{D''}$ for $D', D'' \in m(\bar{u})$, then $z \notin \overline{D}$ for any $D \in m(\bar{u})$, $D' \neq D \neq D''$.

The proof of this fact is based on the following observation. Let $B(0, \epsilon) \subset \mathbb{R}^{N-1}$ and let d be any positive real number. We define the N-dimensional cylinder

$$C = \{z \in \mathbb{R}^N : z = (x, d), \text{ where } x \in B, d \in \mathbb{R}_+\}.$$

For $h = 1, \ldots, H$, let S_h be an affine subspace of $\mathbb{R}^N$ with $\dim(S_h) \leq N - 2$. Then there exists a path $[x, y] \subset C$ such that $[x, y] \cap S_h = \emptyset$ for every h.

The proof. Let s_h be the projection of S_h into $B(0, \epsilon)$. Then $\dim(s_h) \leq N - 2$ and therefore has measure zero in $B(0, \epsilon)$. The countable union of set of measure zero, has measure zero. Thus there exists $x \in B(0, \epsilon)$ such that $x \notin s_h$ for all h. Then, the set $\{(x, d) : d \in \mathbb{R}_+\}$ is the desired path.

Q.E.D.

Theorem 9 reduces the process of verifying whether a piecewise linear mechanism u is an extreme point to determining if there are other mechanisms in the face F_u. In turn this is equivalent to determining whether a consistent system of linear equations has multiple solutions. We expand and illustrate this statement in Subsection 6.3, where we analyze another example.

Before turning to that pursuit, we discuss another application of the faces F_u identified above. Although some extreme points are not piecewise linear, there is no great loss in restricting attention to piecewise linear extreme points of W. This is the content of the following theorem.

Theorem 10 *The set of feasible mechanisms W is the closed convex hull of the set of its piecewise linear, extreme points.*

Proof Let $\bar{u}$ be an extreme point of W that is not piecewise linear, and let $\bar{t}(x) = \nabla\bar{u}(x) \cdot x - \bar{u}(x)$ be its corresponding transfer function. Let $I_n = \{0, 1/n, 2/n, \ldots, n/n\}$. Thus, $(I_n)^N$ is a discretization of the set I^N, increasingly finer as n tends to infinity. For each $x \in I^N$, define $v^n(x) = max_{z\in(I_n)^N}[\nabla\bar{u}(z)\cdot x - \bar{t}(z)]$. It is routine to verify that v^n belongs to W, and that

$$\sup_{x\in I^N} |v^n(x) - \bar{u}(x)| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{9}$$

The mechanism v^n belongs to F_{v^n}, the face of W defined earlier. Note that if e_n^k is an extreme point of F_{v^n}, then it is also an extreme point of W. (To see this, assume by way of contradiction that e_n^k is not an extreme point of W. Then $e_n^k = (1/2)e' + (1/2)e''$ for some e', e'' in W, $e' \neq e^n \neq e''$. Since F_{v^n} is a face, however, e', e'' must then be in F_{v^n}. But then e_n^k is not an extreme point of F_{v^n}.)

The face F_{v^n} is convex and compact; therefore F_{v^n} is the closure of the convex hull of its extreme points (Krein-Milman Theorem). Hence, for each v^n, there is

$$w^n = \sum_{k=1}^{K^n} \alpha_n^k e_n^k$$

where $\alpha_n^k \in (0,1]$; $\sum_{k=1}^{K^n} \alpha_n^k = 1$; for each $1 \leq k \leq K^n$, e_n^k is a piecewise linear extreme point of W; and

$$\|v^n - w^n\|_\infty \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{10}$$

Combining (9) and (10) it follows that $\|w^n - \bar{u}\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$. *Q.E.D.*

Since the closure of the set of extreme points of W is the minimal closed subset of W whose convex closure equals W (Schaefer (1968), Corollary to Theorem 10.5, page 68); we have the following result.

Corollary 2 *The set of piecewise linear extreme points of W is norm dense in the set of extreme points of W.*

6.2 Another Example

The following example identifies an extreme point in which randomization occurs over all goods for all consumers within a market segment.

Example 3 *Mixing on all goods. Let $N = 2$ and let $u \in W$ be defined by*

$$u(x) = \max\{0,\ (0.4x_1 + 0.6x_2 - \frac{1}{5}),\ (x_1 + x_2 - \frac{3}{5})\}.$$

The graph of u and its market segments $\{A^j\}_{j=0}^2$ are depicted in Figure 6.

To see that u is indeed an extreme point, suppose temporarily that it is not. By Theorem 7, $u = \frac{1}{2}u_1 + \frac{1}{2}u_2$ where u_1 and u_2 are piecewise linear and belong to W. Furthermore, the market segments $\{A^j\}_{j=0}^2$ determined from u suffice to define the linear pieces of u_1 and u_2. Note also that

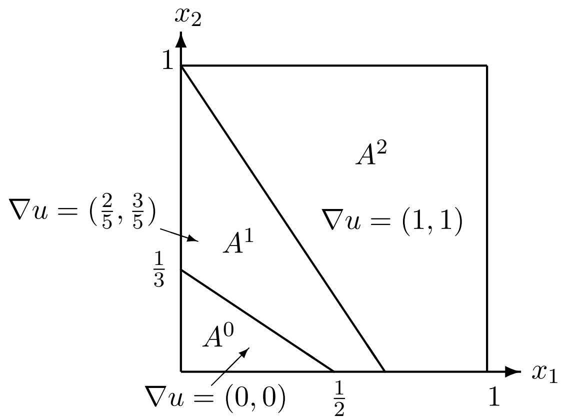

Figure 6: Mixing in all Goods

∇u must be the average of ∇u_1 and ∇u_2. Thus, for $i = 1, 2$, ∇u_i must be $(0, 0)$ in A^0, and ∇u_i must be $(1, 1)$ in A^2. Pick any i in $\{1, 2\}$. It follows that

$$u_i(x) = \begin{cases} (0,0) \cdot (x_1, x_2) - 0, & \text{if } x \in A^0 \\ (c_1, c_2) \cdot (x_1, x_2) - c_0, & \text{if } x \in A^1 \\ (1,1) \cdot (x_1, x_2) - b_0, & \text{if } x \in A^2 \end{cases}, \tag{11}$$

for some $b_0, c_0 \in [0, \infty)$ and $c_1, c_2 \in [0, 1]$. The value of these unknowns is determined by the boundaries of the market segments, i.e., $A^0 \cap A^1$ and $A^1 \cap A^2$. From u, it follows that $A^0 \cap A^1 = \{x \in I^2 : x_2 = \frac{1}{3} - \frac{2}{3}x_1\}$, and $A^1 \cap A^2 = \{x \in I^2 : x_2 = 1 - \frac{3}{2}x_1\}$. From u_i, the boundaries in question are $A^0 \cap A^1 = \{x \in I^2 : x_2 = \frac{c_0}{c_2} - \frac{c_1}{c_2}x_1\}$ and $A^1 \cap A^2 = \{x_2 = \frac{c_0 - b_0}{c_2 - 1} - \frac{c_1 - 1}{c_2 - 1}x_1\}$. We thus obtain the following system of four equations and four unknowns: $\frac{c_0}{c_2} = \frac{1}{3}$, $\frac{c_1}{c_2} = \frac{2}{3}$, $\frac{c_0 - b_0}{c_2 - 1} = 1$, $\frac{c_1 - 1}{c_2 - 1} = \frac{3}{2}$. The unique solution to the system is $c_0 = \frac{3}{5}$, $c_1 = \frac{2}{5}$, $c_2 = \frac{3}{5}$, and $b_0 = 1$. Thus, u_i equal u, a contradiction that proves u is an extreme point.

Remark 7 *The mechanism $\bar{u}$ (with transfer $\bar{t}$) is undominated. To see this, suppose $\bar{t}$ is dominated by t derived from a mechanism u. Since $I^2 \subset \bigcup_{x \in A^0} \mathcal{R}_x$, then $u \geq \bar{u}$ (Theorem 4). Therefore u must be equal to $\bar{u}$, a contradiction.*

The example shows that mixing in all goods may be a feature of the optimal mechanism.

The argument used to prove that $\bar{u}$ is an extreme point is an application of Theorem 7. In the next section we describe an alternative procedure to determine whether a candidate mechanism is an extreme point.

6.3 Identifying Extreme Points

We will describe an algebraic method to determine whether any proposed piecewise linear mechanism u is an extreme point. Based on this method, we argue that piecewise linear extreme points with randomization are plentiful.

A given piecewise linear mechanism $\bar{u}$ is an extreme point if and only if $F_{\bar{u}}$ is the singleton $\{\bar{u}\}$ (Theorem 9). If there is $u \neq \bar{u}$ in $F_{\bar{u}}$, then u and $\bar{u}$ must define the same market segments. Market segments are determined by finitely many linear inequalities. The collection of boundaries between adjacent market segments constitute a system of linear equations. Any mechanism u in $F_{\bar{u}}$ is a solution to that system of linear equations. These ideas will be developed presently.

Let $\bar{u} \in W$ be a piecewise linear mechanism defining market segments $\{A_j)_{j=0}^{J}$. Suppose there is $u \neq \bar{u}$ in $F_{\bar{u}}$. There is no loss of generality in requiring, in addition, that both mechanisms define the exact same market segments (i.e. $m(u) = m(\bar{u})$). (Because $u \in F_{\bar{u}}$, the market segments it defines are coarser than those defined by $\bar{u}$. Then, any strict convex combination between u and $\bar{u}$ yields a mechanism defining exactly the same market segments as $\bar{u}$.) We may therefore express u using the market segments defined by $\bar{u}$

$$\text{for } j = 1, \dots J,\ x \in A^j \implies u(x) = \nabla u^j \cdot x - t_u^j.$$

Suppose the market segments A^j and A^{j-1} are adjacent, in the sense that their common boundary $\overline{A^j} \cap \overline{A^{j-1}}$ is an $(N-1)$-dimensional set (See Remark 6). Since $\bar{u}$ and u define the same market segments, they must define the same boundaries. Therefore, we have

$$\overline{A^j} \cap \overline{A^{j-1}} = \{x \in I^N : \nabla \bar{u}^j \cdot x - t_{\bar{u}}^j = \nabla \bar{u}^{j-1} \cdot x - t_{\bar{u}}^{j-1}\} = \{x \in I^N : \nabla u^j \cdot x - t_u^j = \nabla u^{j-1} \cdot x - t_u^{j-1}\}.$$

Thus, it must be the case that $\nabla u^j \cdot x - t_u^j = \nabla u^{j-1} \cdot x - t_u^{j-1} \iff \nabla \bar{u}^j \cdot x - t_{\bar{u}}^j = \nabla u^{j-1} \cdot x - t_{\bar{u}}^{j-1}$. In turn this implies that

$$\exists \alpha^j \in \mathbb{R} \ : \ \nabla u^j - \nabla u^{j-1} = \alpha^j (\nabla \bar{u}^j - \nabla \bar{u}^{j-1}). \tag{12}$$

To build the system of linear equations, one for each boundary, we use a convenient listing of the boundaries. Suppose to that end that the market segments $\{A^j\}_{j=0}^{J}$ can be labeled so that

(i) $A_0 = \{x \in I^N : \nabla \bar{u}(x) = \mathbf{0}\}$,

(ii) $A_J = \{x \in I^N : \nabla \bar{u}(x) = \mathbf{1}\}$, and

(iii) for $1 \leq j \leq J$, $\overline{A}^j \cap \overline{A}^{j-1}$ is $N-1$ dimensional.

The first two requirements guarantee that the problem is relevant: if either (i) or (ii) do not hold, then $\bar{u}$ is dominated and therefore it is unlikely to be chosen by the seller. Condition (iii) states that market segments may be reordered so that the boundaries between contiguous (according to j) market segments are non-empty $(N-1)$-dimensional sets. The market segments in Examples 1 and 3 have been labeled to satisfy these properties. We will use them as our canonical examples throughout this section. Condition (iii) is mainly for convenience. It permits us to index the relevant boundaries to build the system of equations. If (iii) did not hold, the resulting system of equations would vary, but similar arguments would still apply.

Because of (iii), each j represents a boundary. Therefore, there is one equation (12) for each j, that is to say $\nabla u^j = \nabla u^{j-1} + \alpha^j(\nabla \bar{u}^j - \nabla \bar{u}^{j-1})$, for $j = 1, 2, \ldots, J$. Reordering terms and substituting repeatedly, we may write for $j = 1, 2, \ldots, J$

$$\nabla u^j = \nabla u^0 + \sum_{k=1}^{j} \alpha^k (\nabla \bar{u}^k - \nabla \bar{u}^{k-1}).$$

Since $u \in F_{\bar{u}}$ and $\nabla \bar{u}^0 = \mathbf{0}$, then $\nabla u^0 = \mathbf{0}$. This simplifies the system farther: for $j = 1, 2, \ldots, J$

$$\nabla u^j - \sum_{k=1}^{j} \alpha^k (\nabla \bar{u}^k - \nabla \bar{u}^{k-1}) = 0. \tag{13}$$

The unknowns in system (13) are $\{\nabla u^j\}_{j=1}^{J-1}$ and $\{\alpha^j\}_{j=1}^{J}$. System (13) is consistent; a trivial solution is $\alpha^j = 1$, and $\nabla u^j = \nabla \bar{u}^j$ for all j. *If system (13) has no additional (linearly independent) solution then the mechanism $\bar{u}$ is an extreme point.*

We illustrate this discussion considering a two-goods case (i.e., $N = 2$), and a mechanism $\bar{u}$ with three pieces (i.e., $J = 2$) satisfying $(i) - (iii)$ above. This is the case in Examples 1 and 3. (Recall that since $\nabla \bar{u}^0 = \mathbf{0}$ and $\nabla \bar{u}^J = \mathbf{1}$, any u in $F_{\bar{u}}$ must also satisfy $\nabla u^0 = \mathbf{0}$ and $\nabla u^J = \mathbf{1}$, and that for each j, ∇u^j is an element of I^N and α^j is an element of $\mathbb{R}$.) System (13) then becomes

$$\begin{pmatrix} 1 & 0 & -\nabla \bar{u}_1^1 & 0 \\ 0 & 1 & -\nabla \bar{u}_2^1 & 0 \\ 0 & 0 & \nabla \bar{u}_1^1 & (1 - \nabla \bar{u}_1^1) \\ 0 & 0 & \nabla \bar{u}_2^1 & (1 - \nabla \bar{u}_2^1) \end{pmatrix} \cdot \begin{pmatrix} \nabla u_1^1 \\ \nabla u_2^1 \\ \alpha^1 \\ \alpha^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

where supraindices indicate the market segment j and subindices the good i.

If the coefficient matrix has full rank, then the trivial solution $u = \bar{u}$ (i.e., $\nabla u^j = \nabla \bar{u}^j$ for all j) is the only solution, and $\bar{u}$ is an extreme point. In both Examples 1 and 3 the coefficient matrix has full rank. Thus, the mechanisms in those examples are extreme points; we have just provided another proof.

The rank of the coefficient matrix is fully determined by its last two rows. The matrix has full rank except when the rows $(0, 0, \nabla \bar{u}_1^1, (1 - \nabla \bar{u}_1^1))$ and $(0, 0, \nabla \bar{u}_2^1, (1 - \nabla \bar{u}_2^1))$ are not linearly independent. Linear dependence can only arise if $\nabla \bar{u}_1^1 = \nabla \bar{u}_2^1$. The preceding argument shows that, generically, when $N = 2$, the piecewise linear mechanisms with three pieces and $\nabla \bar{u}^0 = \mathbf{0}$ and $\nabla \bar{u}^2 = \mathbf{1}$ are extreme points. Geometrically, one such mechanism $\bar{u}$ is *not* an extreme point if and only if the boundaries between market segments are parallel lines in I^2.

The examples discussed are canonical in the following sense. For each j there are N equations, one for each good. The last N equations correspond to the boundary between market segments A^J and A^{J-1}. Since $\nabla u^J = \nabla \bar{u}^J = \mathbf{1}$, the last N equations constitute a linear system with the J unknowns $\{\alpha^j\}_{j=1}^{J}$ as the only unknowns. Provided $J \leq N$, the same arguments made above hold. Note that J is the number of market segments minus one, or the number of market segments where some assignment is made, according to the mechanism $\bar{u}$. Thus, provided that number is no larger than the number of goods, extreme points are abundant.

Even if the system of equations (13) has multiple solutions, we cannot immediately conclude that $\bar{u}$ is not an extreme point. For this to be the case, the non-trivial solution must be such that the resulting u is in W. In other words, for all j, the vector ∇u^j must belong to I^N. It is possible (and simple) to construct an extreme point with two goods and 4 linear pieces such that the corresponding system (13) has multiple solutions. Still none of the non-trivial solutions is feasible in the sense discussed.

7 Odds and Ends

We conclude with some final remarks.

1. For any given prior density of buyer's valuations, there may be multiple solutions to the seller's problem. Some mechanisms, however, are the unique solution to the seller's problem for some given density of buyer's valuations. These mechanisms have a desirable robustness quality: small variations in prior beliefs will not change significantly the optimal mechanism. Formally, the mechanisms mentioned are a subset of the exposed points of the feasible set. An element $\bar{t} \in T$ is an exposed point of T if there exists a continuous linear functional f such that $\langle f, \bar{t} \rangle = 0$ and $\langle f, t \rangle < 0$ for all t in $T \setminus \{\bar{t}\}$. Intuitively, the a mechanism $\bar{t}$ is an exposed point if there is a hyperplane strictly supporting T at $\bar{t}$, i.e., the intersection of T with the supporting hyperplane is the singleton $\{\bar{t}\}$.

   The definition of exposed point, however, places no additional restriction on f. In particular, f could take both positive and negative values. We are interested in identifying the exposed points where the supporting functional f is non-negative, more precisely a density function. We explore this question in a companion paper. Let $\bar{t}$ be the transfer function corresponding to a piecewise linear, extreme point of W. In Manelli and Vincent (2004b) we prove that if $\bar{t}$ is sequentially undominated (and the market segments satisfy an additional condition) then $\bar{t}$ is the unique solution to the seller's problem for some density of buyer's valuations. We also conjecture that every undominated, piecewise linear extreme point of W is a unique solution to the seller's problem for some prior density of valuations.

2. Much of our analysis can incorporate some form of production costs.[6] The objective function in the seller's problem is $E[\nabla u(x) \cdot x - u(x)]$. Let $C : I^N \longrightarrow \mathbb{R}$ be quasi-concave. Then, $E[\nabla u(x) \cdot x - u(x) - C(\nabla u(x))]$ is quasi-convex and it achieves a maximum at an extreme point of W. The introduction of the cost function C is more natural in a reinterpretation of the formal model: the seller produces a single good but must decide on the good's characteristics, an N-dimensional vector; the buyer buys at most one unit of the commodity and the buyer's

[6] We are grateful to Kim Border for this observation.

private information x is the buyer's valuation for the different characteristics. In this context, $p(x) = \nabla u(x)$ is the characteristics selected by the seller given the buyer's reported valuations x. The function $C(\nabla u(x))$ thus represents the cost of providing the level of characteristics $\nabla u(x)$ (Border (2002)).

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9 Appendix

Lemma 1, presented without proof, collects several properties of IC, IR mechanisms.

Lemma 1 *If u satisfies IC and IR then (i) $u(x)$ is non-negative for all x; (ii) u is non-decreasing: $x' \geq x \implies u(x') \geq u(x)$; (iii) u is continuous; (iv) u is a.e. differentiable; and (v) u is monotone: $(\nabla u(x') - \nabla u(x)) \cdot (x' - x) \geq 0$, for all x', x.*

Lemma 2 *Let $W = \{u \in C^0(I^N) \mid u(x)$ is convex, $\nabla u(x) \in I^N$ a.e., and $u(\mathbf{0}) = 0\}$. The set W is compact with respect to the sup norm.*

Proof The family of functions W is equicontinuous and uniformly bounded. The Arzela-Ascoli Theorem implies the desired result.

Q.E.D.

Lemma 3 *Let $W = \{u \in C^0(I^N) \mid u(x)$ is convex, $\nabla u(x) \in I^N$ a.e., and $u(\mathbf{0}) = 0\}$. Let $u \in W$ and let t map $x \mapsto \nabla u(x) \cdot x - u(x)$. For $n = 1, 2, \ldots$, let u^n be an element of W and t^n map $x \mapsto \nabla u^n(x) \cdot x - u^n(x)$. If the sequence $\{u^n\}$ converges uniformly to $u \in W$, then,*
(i) $\{\nabla u^n\} \xrightarrow{\lambda-a.e.} \nabla u$ and therefore $\{\nabla u^n(x) \cdot x\} \xrightarrow{\lambda-a.e.} \nabla u(x) \cdot x$; and
(ii) $\{t^n\} \xrightarrow{L_1} t$, and
(iii) T is compact in the L^1 norm.

Proof (i) For $n = 1, 2, \ldots$, let D^n be the set of x in the interior of I^N where $u^n(x)$ is differentiable, and let D' be similarly defined for u. The sets $D^n, \forall n$ and D' are dense in I^N and have λ-measure one (Rockafellar (1970), Theorem 25.5, page 246). The set $D = (\bigcap_{n \geq 1} D^n) \cap D'$ has full measure.

Pick any $x \in D$. Since u^n is differentiable at x, $\nabla u^n(x)$ equals the unique subgradient at x (Rockafellar (1970), Theorem 25.1, page 242). Therefore

$$\forall y \in \mathbb{R}^N, \frac{u^n(x - \delta y) - u^n(x)}{\delta} \leq \nabla u^n(x) \cdot y \leq \frac{u^n(x + \delta y) - u^n(x)}{\delta},$$

for all $\delta \in (0, \bar{\delta}]$ such that $(x + \bar{\delta} y) \in I^N$ and $(x - \bar{\delta} y) \in I^N$. (Such $\bar{\delta}$ exists because x is in the interior of I^N.)

It follows that for any $\epsilon > 0$ and y, there is $\bar{n}$ such that $n > \bar{n}$ implies

$$\frac{u(x - \delta y) - u(x)}{\delta} - \epsilon \leq \nabla u^n(x) \cdot y \leq \frac{u(x + \delta y) - u(x)}{\delta} + \epsilon. \tag{14}$$

To see this, note that given any two sequence of real numbers r^n, s^n, with $r^n \geq s^n, \forall n$, and $s^n \longrightarrow s$, the following inequalities hold: $r^n - s \geq s^n - s \geq -\|s^n - s\|$. Since for any $\epsilon > 0$, there is $\bar{n}$ such that $n > \bar{n}$ implies $-\|s^n - s\| \geq -\epsilon$, it follows that $[n > \bar{n} \implies r^n - s \geq -\epsilon]$. The same argument can be used to obtain both inequalities in (14).

Finally letting $\delta \downarrow 0$ in (14) and using the definition of a gradient, it follows that $n > \bar{n}$ implies

$$\nabla u(x) \cdot y - \epsilon \leq \nabla u^n(x) \cdot y \leq \nabla u(x) \cdot y + \epsilon.$$

Since y and ϵ are arbitrary, the proof of (i) is complete.

(ii) By (i), $|t^n - t| \xrightarrow{\lambda-a.e.} 0$. By construction $|t^n - t|$ is bounded. The Lebesgue Dominated Convergence Theorem implies that $\int |t^n - t|\, d\lambda \longrightarrow 0$. This completes the proof.

(iii) It follows from (ii) and Lemma 2. *Q.E.D.*

We provide without proof the following well-known result.

Lemma 4 *Let X be a locally convex, topological vector space, W be a non-empty compact, convex subset of X, and $S : W \longrightarrow \mathbb{R}$ be a continuous linear function. Then the set F of maximizers of S over W is a face of W. Furthermore, F contains an extreme point of W.*

Lemma 5 *The set of piecewise linear mechanisms in W is dense in W with the sup norm.*

Proof Sketch. Pick any $u \in W$ and let $t = \nabla u \cdot x - u$ be its corresponding transfer function. Let $I_n = \{0, 1/n, 2/n, \ldots, n/n\}$; I_n^N is a discretization of the set I^N. For each $z \in I_n^N$, define the linear function of $x \in I^N$, $\nabla u(z) \cdot x - t(z)$, and consider the function $v^n(x) = max_{z \in I_n^N} \nabla u(z) \cdot x - t(z)$. It is routine to check that $\sup_{x \in I^N} |v^n(x) - u(x)|$ tends to zero as n tends to infinity. *Q.E.D.*

Lemma 6 *Let u and u' be two mechanisms in W and let t and t' denote their respective transfer functions. Suppose t' dominates t. Then, there exist measurable functions $\nabla u'$ and ∇u both defined from I^N into I^N, such that*

(i) $\nabla u'(x) \in \partial u'(x)$ *and* $\nabla u(x) \in \partial u(x)$, *(where $\partial u(x)$ is the subdifferential of u at x) and*

(ii) $[\nabla u'(x') - \nabla u(x)] \cdot x \geq [u'(x) - u(x)]$ *for all* $x \in I^N$,

(iii) $\forall \delta \in (1, \delta')$, $\;-[u'(\delta x) - u(\delta x)] = \int_\delta^{\delta'} [\nabla u'(\gamma x) - \nabla u(\gamma x)] \cdot x \, d\gamma$,

(iv) $\nabla u'(x) \cdot x - u'(x) \geq \nabla u(x) \cdot x - u(x)$, $\forall x \in I^N$.

Proof Let $D' = \{x \in I^N : \nabla u'(x) \text{ exists}\}$, $D = \{x \in I^N : \nabla u(x) \text{ exists}\}$ and $D'' = \{x \in I^N : [\nabla u'(x) - \nabla u(x)] \cdot x \geq [u'(x) - u(x)]\}$. Since $\lambda(D'') = \lambda(D') = \lambda(D) = 1$, then $\lambda(D'' \cap D' \cap D) = 1$.

Let $E = \{(x, \nabla u'(x), \nabla u(x)) : x \in D'' \cap D' \cap D\}$. Then $E \subset I^N \times I^N \times I^N$ and $\overline{E}$ is compact.

Let $\texttt{proj}_{I^N}(\overline{E}) = \{x \in I^N : (x, y, z) \in \overline{E}\}$; $\texttt{proj}_{I^N}(\overline{E})$ is the projection of $\overline{E}$ on its first coordinate. The set $\overline{E}$ is the graph of the correspondence $\varphi : \texttt{proj}_{I^N}(\overline{E}) \longrightarrow I^N \times I^N$ defined by $\varphi(x) = \{(y, z) \in I^N \times I^N : (x, y, z) \in \overline{E}\}$. By the selection Theorem of Kuratowsky and Ryll-Nardzewski (see for instance Hildenbrand (1974)), there is a measurable selection g of φ.

We first show that $\texttt{proj}_{I^N}(\overline{E}) = I^N$. Suppose not. Then there is $x \in I^N$ and $x \notin \texttt{proj}_{I^N}(\overline{E})$. Since $\texttt{proj}_{I^N}(\overline{E})$ is closed, there is $\epsilon > 0$ such that $B(x, \epsilon) \cap \texttt{proj}_{I^N}(\overline{E}) = \emptyset$ where $B(x, \epsilon) = \{x' \in I^N : \|x' - x\| < \epsilon\}$. Thus $\lambda(\texttt{proj}_{I^N}(\overline{E})) < 1$. By construction $(D'' \cap D' \cap D) \subset \texttt{proj}_{I^N}(\overline{E})$, and therefore $\lambda(\texttt{proj}_{I^N}(\overline{E})) = 1$, a contradiction.

Let $\partial u'$ and ∂u denote the subdifferential correspondence of u' and u respectively. Since both u' and u are convex, for all $x \in I^N$, $\partial u(x)$ is non-empty, and closed.

It is a matter of verifying definitions to show that $g(x) \in (\partial u'(x), \partial u(x))$ for all $x \in I^N$. Abusing notation slightly, we will denote $(\nabla u'(x), \nabla u(x)) = g(x)$ for all $x \in I^N$.

Finally, note that by construction $[\nabla u(x') - \nabla u(x)] \cdot x \geq [u'(x) - u(x)]$ for all $x \in I^N$.

Then, from Krishna and Maenner (2001), Theorem 1, it follows that the integral (5) is valid for any measurable functions satisfying (i) above. Condition (ii) states that $t' \geq t$ everywhere in I^N. *Q.E.D.*

Proof of Theorem 6 For any $A^j \in m(u)$, let ∇u^j denote the gradient of u evaluated at any $x \in A^j$, and t^j be the transfer for every $x \in A^j$. Therefore,

$$u(x) = \max\{\nabla u^j \cdot x - t^j \, : \, A^j \in m(u)\}.$$

First, suppose that for every $A^j \in m(u)$, $\nabla u^j \neq \mathbf{0}$. We will show u is dominated.

Let $\mathcal{M} = \{A^j \in m(u) : \mathbf{0} \in \overline{A^j}\}$. and define

$$\begin{aligned} v(x) &= \max\{\nabla u^j \cdot x - t^j \, : \, A^j \in m(u) \setminus \mathcal{M}\}, \\ w(x) &= \max\{\nabla u^j \cdot x - t^j \, : \, A^j \in \mathcal{M}\}. \end{aligned}$$

The set $\mathcal{M}$ is non-emtpy because $u(\mathbf{0}) = 0$. Suppose momentarily that $m(u) \setminus \mathcal{M}$ is not empty; we will show later in the proof that the alternative case is trivial.

We use the functions v and w to define a new function u', and to express u. For every $x \in I^N$, let

$$u'(x) = \max\{v(x), 0\}. \tag{15}$$

We will show that u' dominates u. Note that for every $x \in I^N$,

$$u(x) = \max\{v(x), w(x), 0\}. \tag{16}$$

The mechanism u has three components and its corresponding transfer t is strictly positive only on the effective domain of v. More precisely, we will show that

$$\begin{aligned} A^j \in \mathcal{M} &\implies t^j = 0, \text{and} \\ A^k \in m(u) \setminus \mathcal{M} &\implies t^k > 0. \end{aligned}$$

To see this, pick any $A^j \in \mathcal{M}$. Observe that $x \in A^j$ and $\mathbf{0} \in \overline{A^j}$ implies that $\alpha x \in A^j$ for any $\alpha \in (0, 1]$. By definition, $u(\alpha x) = \nabla u^j \cdot \alpha x - t^j$. If $t^j > 0$, then there is $\alpha \in (0, 1]$ such that $u(\alpha x) < 0$, a contradiction. We have thus shown that $t^j = 0$.

Pick then any $A^k \in m(u) \setminus \mathcal{M}$. By definition of market segments,

$$\nabla u^j \cdot x - t^j > \nabla u^k \cdot x - t^k, \forall x \in A^j.$$

Let x tend to $\mathbf{0}$ which we may do because $\mathbf{0} \in \overline{A^j}$. Then

$$\nabla u^j \cdot \mathbf{0} - t^j \geq \nabla u^k \cdot \mathbf{0} - t^k.$$

If the expression above is satisfied as equality, then $\mathbf{0} \in \overline{A^k}$, and this is a contradiction since $A^k \in m(u) \setminus \mathcal{M}$. Thus, we conclude that

$$-t^j > -t^k.$$

Since $A^j \in \mathcal{M}$, $t^j = 0$ and thus, we have $t^k > 0$.

Consider now the mechanism u'. It has two components and its corresponding t' is strictly positive also on the effective domain of v.

Observe that, by construction, $\max\{w(x), 0\} \geq 0$ for every $x \in I^N$ and it is strictly positive for any $x \in \mathit{int}\, A^j$, $A^j \in \mathcal{M}$ because the gradient $\nabla u^j \neq 0$. Hence $w(x) > 0$ in a set of positive measure. Thus, the effective domain of v in the definition (15) of u', $\{x \in I^N : v(x) > 0\}$, strictly contains the effective domain of v in the definition (16) of u, $\{x \in I^N : v(x) > \max\{w(x), 0\}\}$. This completes the proof under the assumption that $m(u) \setminus \mathcal{M}$ is non-empty.

If $m(u) \setminus \mathcal{M} = \emptyset$, then u yields revenue $t = 0$ because $A^j \in \mathcal{M}$ implies $t^j = 0$. Hence u is clearly dominated. This completes the proof of the first part.

Second, suppose that $\nabla u^j \neq \mathbf{1}$ for every $A^j \in m(u)$. For any $r \geq 0$ and $x \in I^N$ define the functions

$$\begin{aligned} v_r(x) &= \mathbf{1} \cdot x - [N - u(\mathbf{1}) - r] \\ u_r(x) &= \max\{v_r(x), u(x)\}. \end{aligned}$$

We will prove that for sufficiently small r, u_r dominates u.

Define $K_r = \{x \in I^N : v_r(x) \geq u(x)\}$. For all $x \in I^N \setminus K_r$, $u_r(x) = u(x)$ and therefore both mechanisms generate the same transfer.

Pick any $x \in K_r$. The transfer generated by u_r is

$$\begin{aligned} t_r &= N - u(\mathbf{1}) - r = \mathbf{1} \cdot \mathbf{1} - u(\mathbf{1}) - r, \\ &= [\mathbf{1} - \nabla u(\mathbf{1})] \cdot \mathbf{1} + \nabla u(\mathbf{1}) \cdot \mathbf{1} - u(\mathbf{1}) - r. \end{aligned} \tag{17}$$

If the element $x \in K_r$ belongs to $A^j \in m(u)$, then the transfer generated by u is

$$\begin{aligned} t^j &= \nabla u(x) \cdot x - u(x) \\ &\leq \nabla u(x) \cdot x - u(\mathbf{1}) - \nabla u(\mathbf{1}) \cdot (x - 1), \\ &\leq [\nabla u(x) - \nabla u(\mathbf{1})] \cdot x + \nabla u(\mathbf{1}) \cdot \mathbf{1} - u(\mathbf{1}), \end{aligned} \tag{18}$$

where the inequality follows because of the convexity of u. Subtracting (18) from (17) we obtain

$$\begin{aligned} t_r - t^j &\geq [\mathbf{1} - \nabla u(\mathbf{1})] \cdot \mathbf{1} - [\nabla u(x) - \nabla u(\mathbf{1})] \cdot x - r. \\ &\geq [\mathbf{1} - \nabla u(\mathbf{1})] \cdot (\mathbf{1} - x + x) - [\nabla u(x) - \nabla u(\mathbf{1})] \cdot x - r. \\ &\geq [\mathbf{1} - \nabla u(\mathbf{1})] \cdot (\mathbf{1} - x) + [\mathbf{1} - \nabla u(x)] \cdot x - r. \end{aligned}$$

The combination of the first two terms is strictly positive. Therefore, we conclude that

$$\exists\, r_j > 0 : [0 < r < r_j] \implies t_r > t^j.$$

Let $r' = \min\{r_j : A^j \in m(u)\}$.

We have thus proved that for $0 < r < r'$, t_r dominates t. A contradiction. *Q.E.D.*

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